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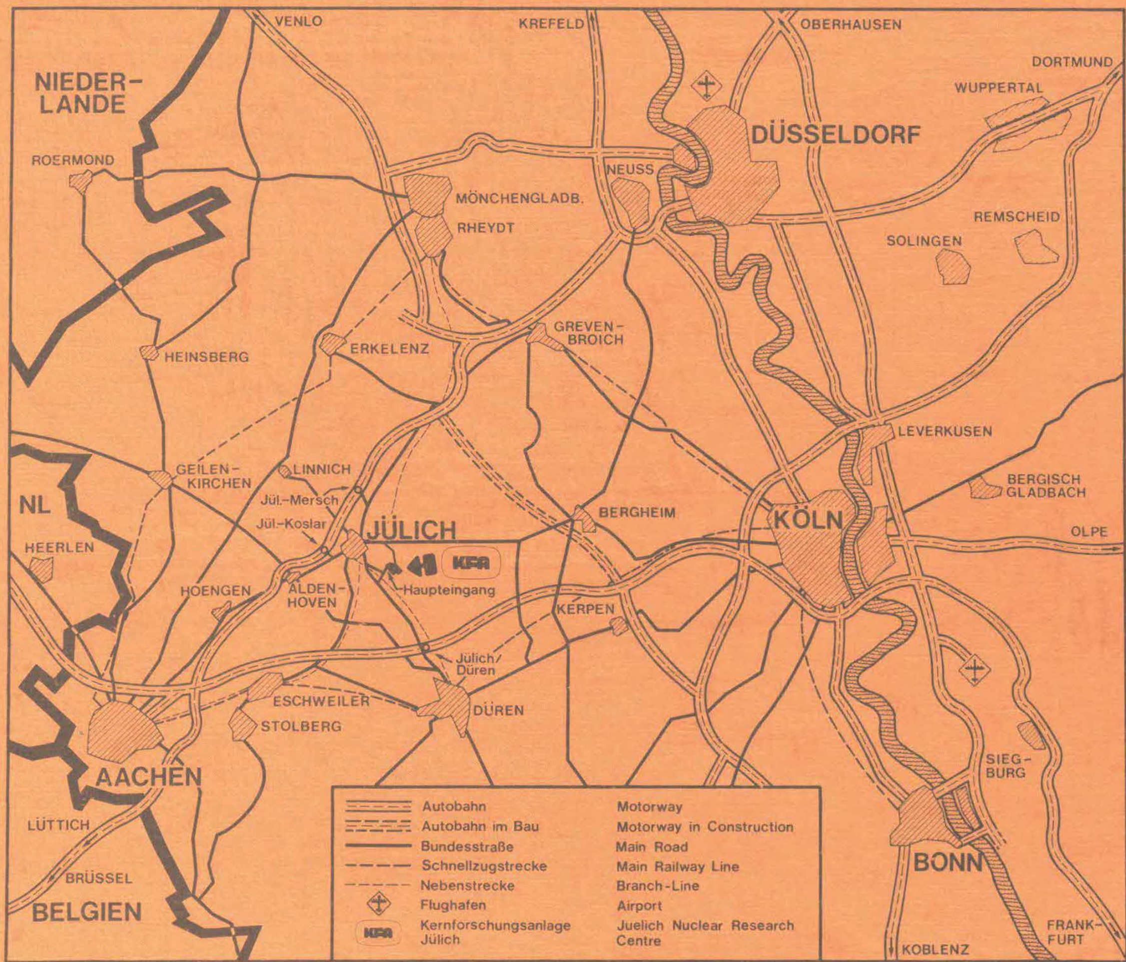

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AXIOMATIC ELECTRODYNAMICS AND MICROSCOPIC MECHANICS*

by

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ABSTRACT

A new approach to theoretical physics, along with the basic formulation of a new MICROSCOPIC MECHANICS for the motion of small charged particles is described in this set of lecture notes. Starting with the classical (Newtonian) mechanics and classical fields, the important but well known properties of Classical Electromagnetic field are discussed up to section 4. The next section describes the usual radiation damping theory and its difficulties. It is argued that the usual treatment of radiation damping is not valid for small space and time intervals and the true description of motion requires a new type of mechanics - the MICROSCOPIC MECHANICS. Section 6 and 7 are devoted to showing that not only the new microscopic mechanics goes over to Newtonian mechanics in the proper limit, but also it is closely connected with Quantum Mechanics.

All the known results of the Schrödinger theory can be reproduced by microscopic mechanics which also gives a clear physical picture. It removes Einstein's famous objections against Quantum Theory and provides a clear distinction between classical and Quantum behavior. Seven Axioms (three on Classical Mechanics, two for Maxwell's theory, one for Relativity and a new Axiom on Radiation damping) are shown to combine Classical Mechanics, Maxwellian Electrodynamics, Relativity and Schrödinger's Quantum Theory within a single theoretical framework under Microscopic Mechanics which awaits further development at the present time.

CONTENTS

1.1 Introduction

These lectures will be divided into two distinct parts. The first part deals with the established theory of classical electrodynamics. But a new axiomatic approach will be described here. The formal development of classical mechanics from Newton's equations to the Lagrangian and Hamiltonian formulations of classical fields will be extended, through Axioms, to the electromagnetic field. That will also bring in the special theory of relativity. We will study some important formal aspects of the classical electrodynamics which is just a classical vector field. This should re-emphasize the fact that fields have many physical attributes like particles. The coupling between particles and fields will involve radiation reaction. The unsolved problems of radiation reaction will be of special interest to us.

A speculative approach to the difficulties of radiation reaction theory will take us to the second part of these lectures. It will reveal a new area of "Microscopic Mechanics" which provides a new but fascinating physical picture for the quantum behavior of small systems. This is neither established nor fully investigated and therefore has difficulties about which no certain answer is available at present. But the physical picture is promising in many ways. Mathematically, one simply has an extended Newton's equation from which one demands stable physical solutions. The condition for existence of such solutions is that a corresponding linear differential equation must exist which will be identified with the Schrödinger's equation. A simple procedure will be followed to reach this connection. Thus, Schrödinger's equation will be '<u>derived</u>' from classical electrodynamics with a single assumption about the radiation reaction force.

We shall show that this is more than just another 'trick' to get the Schrödinger's equation. A proper set of reasons will reveal the logical use of this equation in the same way as practiced in Schrödinger theory. If measurements are done on an ensemble of systems obeying Microscopic Mechanics (which goes over to Newtonian Mechanics in the proper limit), then the results will be exactly given in terms of the probabilities predicted by Schrödinger's theory. The wave function is in fact the probability amplitude. Many speculations will follow thereafter. If this physical picture is true, it will remove Einstein's famous objections against Quantum Theory. We believe that the present day physics is entering an era when one must ask whether we know all the basic concepts and equations of theoretical physics. If not, does Microscopic Mechanics provide a window for looking beyond the closed walls of quantum philosophy?

There are many references for the material (and not the approach for which I have only my unpublished notes) of the first part of the lectures and some of them are:

H. Goldstein, Classical Mechanics (Addison-Wesley, 1962)
F. Rohrlich, Classical Charged Paricles (Addison-Wesley, 1965)
J.D. Jackson, Classical Electrodynamics (John-Wiley, 1963).

The second part needs any standard book on Quantum Mechanics like,

L.I. Schiff, Quantum Mechanics (McGraw-Hill, 1968),

and two papers:

M. Yussouff in Nuovo Cimento 54B, 36(1979)

and Lett. Nuovo Cimento 23, 599(1978).

I will assume the audience to be familiar with the basic concepts of advanced classical Mechanics, elementary electromagnetic theory and elementary quantum theory.

1.2 Classical Mechanics

Let us quickly review some of the formal aspects of classical Mechanics. Starting with the Newton's Axioms (as the Newton's Laws are called by Sommerfeld), especially

$$\frac{d}{dt}(\vec{p}) = \vec{F}_{ext} \tag{1.2-1}$$

for a particle with momentum $\vec{p}$ under the action of the external applied force $\vec{F}_{ext}$, (potential V) one develops the mechanics of many particles under constraints. Then one has the generalized co-ordinates q_j and Lagrangian L, along with the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad \ldots\ldots \tag{1.2-2}$$

or in terms of the generalized momenta $p_j = \frac{\partial L}{\partial \dot{q}_j}$ and the Hamiltonian $H = \Sigma p_j \dot{q}_j - L$, one gets Hamilton's canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{1.2-3}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{1.2-4}$$

We will soon discuss the corresponding equations for the classical fields. Later on, we will refer to the Hamilton-Jacobi theory. The Hamilton-Jacobi equation with the time independent H for a single particle is

$$\frac{\partial}{\partial t} S(q,P,t) = H(q, \frac{\partial W}{\partial q}) \qquad (1.2\text{-}5)$$

where $S(q,P,t)$ is the action function $(=\int p dq)$ which may be written as

$$S(q,P,t) = W(q,P) - Et. \qquad (1.2\text{-}6)$$

Here $W(q,P)$ is called the characteristic function. As the particle moves in the real space, an equivalent movement of constant S surfaces takes place in the configuration space (Goldstein 9-8). The time development of the system can also be described by the motion of these fictitious but mathematically equivalent waves. The velocity u of these waves is related to the velocity v of the particle of mass m and energy E by

$$u = E / (mv) \qquad (1.2\text{-}7)$$

What is the corresponding wave equation? Naturally, it should have been

$$\nabla^2 \phi - \frac{1}{u^2} \frac{d^2 \phi}{dt^2} = 0, \qquad (1.2\text{-}8)$$

but these waves <u>do not undergo diffraction</u>.

In other words, the amplitude of ϕ varies slowly so that in the geometrical optics (eikonal) approximation, equation (1.2-8) may be replaced by

$$(\frac{\partial W}{\partial q})^2 = 2\ m\ (E - V) \quad , \qquad (1.2\text{-}9)$$

which is just the Hamilton-Jacobi equation (1.2-5).

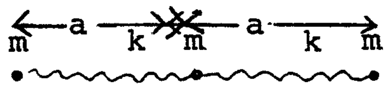
(Homework using Goldstein, ch. 9-8 :) It can be easily shown, for one dimensional motion, that $\phi^* \phi$ = const/(p) = classical probability density.

This is the story for Newtonian Mechanics whose basic equation is (1.2-1). We shall see that for Microscopic Mechanics, the initial equation is modified. Then the corresponding (fictitious) equivalent waves will undergo diffraction and their wave equation is the Schrödinger's equation.

1.3 Classical Fields

The concept of a field is very important in the study of electrodynamics. Physically, a field represents some kind of an 'excitation' present at all points of a spacetime domain. The mathematical idealization of this physical situation is the existence of some well defined functions $\psi_\alpha(x_i,t)$ of co-ordinates x_i and time t. We shall throughout use $x_\mu = (x_i,\ ict)$ so that $\psi_\alpha\ (x_i,t) = \psi_\alpha\ (x_\mu)$. These functions, henceforth called Field Variables, are supposed to represent a 'measure' of the physical excitations. An immediate consequence of such physical considerations is that the description of the field should be independent of co-ordinate systems in space. This restricts the transformation properties of $\psi_\alpha\ (x_\mu)$ under co-ordinate transformations. Then it is possible to classify the fields into two basic categories: the tensor fields and the spinor fields. We discuss tensor fields only. If only one $\psi(x_\mu)$ characterizes the field, then it is a scalar field if it remains invariant under rotations and inversions of co-ordinate axis, and a speudo-scalar field if it changes sign under inversion. The numerical value of scalar ψ at a given space time point remains the same, no matter how the co-ordinates of the space are expressed. A vector field in space has $\psi_\alpha(x_\mu)$, $\alpha = 1,2,3$ which transform like a vector. The numerical value of $\psi_\alpha(x_\mu)$ for fixed spacetime point may change in addition to the change in the functional form when the co-ordinate axes are changed. Similar definitions hold for tensor fields.

It follows from the definition of the field that it ought to behave as a physical entity with most of its characteristics. In fact, the field has energy, momentum and angular momentum; just like any other mechanical system and these attributes will be shown to originate from homogeneity and isotropy of space-time. It is indeed equivalent to a continuous mechanical system with many degrees of freedom. To illustrate the basic aspects of such properties , consider a linear chain of similar particles of mass m joined together with massless springs of length a and force constants k.



If the displacement of i^{th} particle from its equilibrium position is η_i (the motion being confined to one dimension), the small oscillations are described by the Lagrangian

$$L = \frac{1}{2} \sum_i \{m\dot{\eta}_i^{\,2} - k(\eta_{i+1}-\eta_i)^2\} \qquad (1.3\text{-}1)$$

and the equations of motion are

$$m\dot{\eta}_i - k(\eta_{i+1}-\eta_i)+k(\eta_i-\eta_{i-1}) = 0 \qquad (1.3\text{-}2)$$

Now let us approximate this system to a continuous system by letting $a \to 0$ in such a fashion that $\lim\limits_{a\to 0}(\frac{m}{a}) = \mu$ = constant and $\lim\limits_{a\to 0}(ka) = Y$ = const. The physical picture in this approximation is that as the springs become infinitely small, k increases infinitely to keep ka equal to the Young's modulus of the continuous string whereas m decreases to zero so as to keep the mass per unit length constant. Then

$$L = \sum_i aL_i = \int dx\, \mathcal{L} \qquad (1.3\text{-}3)$$

where $$L_i = \frac{1}{2}\left[\mu\dot{\eta}_i^{\,2} - Y\left(\frac{d\eta_i}{dx}\right)^2\right] \qquad (1.3\text{-}4)$$

so that $$\mathcal{L} = \frac{1}{2}\left[\mu\,\dot{\eta}^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]$$

and the equation of motion (1.3-2) becomes

$$\mu\frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0 \tag{1.3-5}$$

Here we have used the fact that as $a \to 0$, $\frac{\eta_{i+1} - \eta_i}{a} \to \frac{d\eta}{dx}$ and summation over i is replaced by integration over the position co-ordinate x. Thus the position co-ordinate x simply replaces the discrete index and is <u>not a generalized co-ordinate</u> of the problem. The generalized co-ordinates η_i become infinite in number and are given by the values of $\eta(x)$ at each point x. Now this $\eta(x)$ obviously describes a scalar field in one dimension in confirmity with the formal definitions. Two important points emerge from these considerations. One is the existence of a Lagrangian density given by (1.3-4) for the field and the other is the fact that the field variable η satisfies an equation of motion. These results are true even in absence of a mechanical model and the equation of motion is then designated as the 'field equation '. Generalizations to three dimensional space and a large number of field variables are trivial and the Lagrangian will have the form

$$L = \int d3x\ \mathcal{L} \tag{1.3-6}$$

where the Lagrangian density $\mathcal{L}$ will involve the field variables η_α, $\alpha = 1, N$ and the $\dot{\eta}_\alpha$, grad(η_α) in general. The equations of motion are then derivable from a variational calculation demanding that the integral

$$I = \int d3xdt\ \mathcal{L} \tag{1.3-7}$$

be extremum. The resulting equation of motion will be

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}_\alpha}\right) + \sum_{k=1}^{3}\frac{d}{dx_k}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta_\alpha}{\partial x_k}\right)}\right) - \frac{\partial\mathcal{L}}{\partial\eta_\alpha} = 0, \quad \alpha=1\ldots N \tag{1.3-8}$$

If the $\mathcal{L}$ is known for a given field, then the field equations are given by equation (1.3-8) but more often the reverse is the case and one tries to 'guess' at the Lagrangian density knowing the field equations satisfied by the field variables. It is useful to note that the field variables behave essentially as a set of generalized co-ordinates of a mechanical system and hence in general many sets of field components can serve to give a convenient formulation of the field just as many sets of generalized co-ordinates may be used for solving the same mechanical problem. Thus the choice of a particular set of field variables is a matter of mathematical convenience.

It is also interesting and very useful to note that the entire formal structure of classical mechanics can be incorporated into the field formulation. Specifically, the Hamiltonian formulation can be applied to the fields in a straightforward manner. Thus one can defines a Hamiltonian density $\mathcal{H}$ by

$$\mathcal{H} = \sum_{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\alpha} \dot{\eta}_\alpha - \mathcal{L} \qquad (1.3\text{-}9)$$

and the Hamiltonian of the system is given by

$$H = \int d^3x \ \mathcal{H}$$

The momentum density is defined as $\pi_\alpha = \dfrac{\partial \mathcal{L}}{\partial \dot{\eta}_\alpha}$, (1.3-10) and satisfies the equation of motion in the Hamiltonian form

$$-\dot{\pi}_\alpha = \frac{\partial \mathcal{H}}{\partial \eta_\alpha} - \sum_k \frac{d}{dx_k} \left(\frac{\partial \mathcal{H}}{\partial (\partial \eta_\alpha / \partial x_k)} \right) \qquad (1.3\text{-}11)$$

Finally the foregoing discussions indicate that the field can be looked upon as a medium through which disturbances can be propagated. The 'excitations' of the field can carry the disturbance from one point to another in space and the process can be depicted by the time dependent solutions of the field equations. Thus the field equations govern the transmission of 'disturbances' through the field and this helps one to do away with the action at a distance point of view. Indeed, whenever two physical entities interact, the interaction propagates through a suitable field. Every observable particle in nature must interact via some kind of a field and hence all physical particles will be surrounded by fields. We will discuss this point of view further in the second part of these lectures. The recognition of the existence of an electromagnetic field and its field equations constitute the starting point of axiomatic electrodynamics.

2.1. Axioms for Electrodynamics

Every axiomatic theory starts with some undefined or self-evident ideas. We take the elementary notions of charged particles and currents as self evident. The space surrounding electrical charges and currents is in a state of excitation. This excitation can be described by the electric field vector $\vec{E}(x_i,t)$ and magnetic field vector $\vec{B}(x_i,t)$. The usual definitions of $\vec{E}$ and $\vec{B}$ are in terms of forces $\vec{F}_e(x_i,t)$ and $\vec{F}_m(x_i,t)$ experienced by a test charge q whose instanteneous velocity is $\vec{v}$. Thus the equations $\vec{F}_e = q\vec{E}$ and $\vec{F}_m = q\vec{v} \times \vec{B}$ define $\vec{E}$ and $\vec{B}$ in the limit of $q \to o$. We will use RATIONALIZED MKS UNITS throughout these lectures. (Also $\mu_o = 4\pi \times 10^{-7}$, $\varepsilon_o = 10^7/(4\pi c^2)$ Farads/meter where c is the speed of light in vacuo). Finally, the charge density $\rho(x_i,t)$ and current density $\vec{j}(x_i,t)$ are defined with respect to a volume which is very small macroscopically and large enough on a microscopic scale to avoid singularities. They are related by the equation of continuity

$$\nabla \cdot \vec{j}(x_i,t) + \frac{\partial \rho(x_i,t)}{\partial t} = o. \tag{2.1-1}$$

Now, the axioms for electrodynamics may be stated as follows:

<u>AXIOM I:</u> Charges and Currents interact via the electromagnetic field characterized by $\vec{E}(x_i,t)$ and $\vec{B}(x_i,t)$.

<u>AXIOM II:</u> The field equations for the electromagnetic field are the Maxwell's equations:

$$\nabla \cdot \vec{E}(x_i,t) = \frac{1}{\varepsilon_o} \rho(x_i,t), \tag{2.1-2}$$

$$\nabla \cdot \vec{B}(x_i,t) = 0 \tag{2.1-3}$$

$$\nabla \times \vec{E}(x_i,t) + \frac{\partial \vec{B}}{\partial t}(x_i,t) = 0, \tag{2.1-4}$$

and

$$\nabla \times \vec{B}(x_i,t) - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_o \vec{j}(x_i,t). \qquad (2.1\text{-}5)$$

The experimental basis for these equations must be provided. This being elementary, we simply state that equation (2.1-2) can be integrated to give Gauss's law: $\oint \vec{E} \cdot d\vec{s} = q/\varepsilon_o$ (and hence Coulomb's law) which is experimentally true. The non-existence of magnetic monopoles justify equation (2.1-3) in the form $\oint \vec{B} \cdot d\vec{s} = o$. Faraday's experimental law of induction is the basis for equation (2.1-4) in the form $\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t}\oint \vec{B} \cdot d\vec{s}$. Finally, equation (2.1-5) is connected with the experimental Biot-Servart law or the Ampere's law for stationary $\vec{E}$. But as we know, the vast amount of experimental checks on the deductions form Maxwell's equations (including non-linear phenomena) provide the real physical foundation for these equations.

2.2 The Potentials and Gauges:

We have seen before that the field variables are analogous to the generalized co-ordinates of a mechanical system. Therefore, one can describe the same field by different sets of field variables. The $\vec{E}$ and $\vec{B}$ description of the electromagnetic field is not the most convenient one. The potentials provide an alternative convenient description of the electromagnetic field. It follows from equation (2.1-3) that $\vec{B}$ can be written as Curl of another vector. Thus

$$\vec{B}(x_i,t) = \nabla \times \vec{A}(x_i,t) \quad . \qquad (2.2\text{-}1)$$

This automatically satisfies equation (2.1-3).

Then it follows from equation (2.1-4) that $\nabla X\vec{E} = -\frac{\partial}{\partial t}(\nabla X\vec{A}) = \nabla X(-\frac{\partial\vec{A}}{\partial t})$ and therefore $\vec{E}$ can differ from $-\frac{\partial\vec{A}}{\partial t}$ by at most a gradient of an arbitrary function $\emptyset$. Thus, if we write

$$\vec{E} = -\nabla\emptyset - \frac{\partial\vec{A}}{\partial t} \qquad (2.2\text{-}2)$$

where $\emptyset$ is a function of space and time, equation (2.1-4) is also satisfied. Now the two other equations can be used to determine $\vec{A}$ and $\emptyset$ if ρ and $\vec{j}$ are specified. Thus we get from equation (2.1-2)

$$\nabla^2\emptyset + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\frac{\rho}{\varepsilon_o}, \qquad (2.2\text{-}3)$$

and equation (2.1-5) yields

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \nabla(\nabla\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\emptyset) - \mu_o\vec{j} \qquad (2.2\text{-}4)$$

Equations (2.2-3) and (2.2-4) can be solved for $\vec{A}$ and $\emptyset$ when ρ and $\vec{j}$ are specified and then using equations (2.2-1) and (2.2-2), $\vec{E}$ and $\vec{B}$ can be determined. $\vec{A}$ is called the vector potential and $\emptyset$ is called the scalar potential. However the question arises whether for given $\vec{E}$ and $\vec{B}$, the new functions $\vec{A}$ and $\emptyset$ are unique. That they are not unique can be seen by considering the following set of potentials

$$\vec{A}' = \vec{A} + \nabla\Lambda$$
$$\emptyset' = \emptyset - \frac{\partial\Lambda}{\partial t} \qquad (2.2\text{-}5)$$

where Λ is an arbitrary function of space and time. The fields derived from there potentials are

$$\vec{E}' = -\nabla\phi' - \frac{\partial\vec{A}'}{\partial t} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$$

and $$\vec{B}' = \nabla X\vec{A}' = \nabla X\vec{A} = \vec{B} \text{ since } \nabla X(\nabla\Lambda) = o \quad .$$

Thus $\vec{A}'$ and ϕ' give rise to the same $\vec{E}$ and $\vec{B}$ as $\vec{A}$ and ϕ, and the two sets are related through an arbitrary function. This implies that the choice of $\vec{A}$ and ϕ is not unique and that this freedom in the choice of Λ can be used to simplify the equations to a great extent. The transformation from $(\vec{A},\phi)$ to $(\vec{A}',\phi')$ is called a Gauge transformation of the second kind and Λ is called the Gauge function. Restricting this function in difference ways, we get different gauges for the potentials but the field calculated from potentials in any gauge is always the same.

Thus, if one restricts $\Lambda(x_i,t)$ to be the solution of the equation

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = o \, , \qquad (2.2\text{-}6)$$

then $$\nabla\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t} = \nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$$

$$= \nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

and without loss of generality this can be equated to zero. Therefore at least all the potentials satisfying the <u>Lorentz condition</u>

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = o \qquad (2.2\text{-}7)$$

must have their gauge function satisfy equation (2.2-6) and are said to belong to the Lorentz gauge. In this gauge, by virtue of equation (2.2-7), equations (2.2-3) and (2.2-4) become

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = - \frac{\rho}{\varepsilon_o} = - \mu_o c^2 \rho \qquad (2.2\text{-}8)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = - \mu_o \vec{J} \qquad (2.2\text{-}9)$$

Thus, if one works with potentials confined to Lorentz gauge (satisfying Lorentz condition) then one is led to the wave equations (2.2-8) and (2.2-9) to be solved for $\vec{A}$ and ϕ. This useful Gauge is very handly in classical electrodynamics but leads to serious trouble in quantum electrodynamics.

Another important gauge is the one where Λ is a solution of the Laplace's equation

$$\nabla^2 \Lambda = o \;, \qquad (2.2\text{-}10)$$

so that
$$\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \Lambda = \nabla \cdot \vec{A}$$

and this can be chosen to be equal to zero without loss of generality. Thus when

$$\nabla \cdot \vec{A} = o \;, \qquad (2.2\text{-}11)$$

is satisfied by chosing Λ to be the solution of Laplace equation (2.2-10), then one gets the Coulomb gauge which is also called the Transverse Gauge. In this gauge, by virtue of equation (2.2-11) the same equations (2.2-3) and (2.2-4) become

$$\nabla^2 \phi = - \frac{\rho}{\varepsilon_o} \qquad (2.2\text{-}12)$$

and $$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\emptyset) - \mu_o\vec{J} \qquad (2.2\text{-}13)$$

The equation (2.2-12) is the so called Poisson equation. Indeed this is called Coulomb gauge because of equation (2.2-12) which can be arrived at in electrostatics using Coulomb's law.

Lorentz Gauge is the most useful one in the convariant formulation and Coulomb gauge is useful in dealing with static problems. However there is no special compulsion for one or the other and indeed one can invent one's own gauge if one so likes. The fact that the fields do not depend upon the gauges and are thus gauge invariant leads to the conclusion that the physical results should be independent of the gauge for the potential and this invariance leads to important consequences in the quantized theory.

2.3. Maxwell's Equations and Relativity

We start simply by redefining a few quantities and rewriting the Maxwell's equations. As stated earlier, let $x_\mu = (x_i, ict)$ with $\mu = 1,2,3,4$ defining the Minkowski space for an observer with a space co-ordinate system (x_1, x_2, x_3) and time t. All Greek letters take values 1,2,3,4 and repeated indices are summations unless specified otherwise. Define the Four current density

$$\mathcal{J}_\mu = (\vec{J}, ic\rho), \qquad (2.3\text{-}1)$$

and the four gradient $$\partial_\mu \equiv \frac{\partial}{\partial x\mu}, \qquad (2.3\text{-}2)$$

to write the equation of continuity as

$$\partial_\mu \mathcal{J}_\mu = o. \tag{2.3-3}$$

Similarly define the four potenital as

$$A_\nu = (\vec{A}, i\emptyset/c), \tag{2.3-4}$$

to get the Lorentz condition

$$\partial_\mu A_\mu = o. \tag{2.3-5}$$

The Maxwell's equations in the Lorentz Gauge may then be written as

$$\partial_\mu \partial_\mu A_\nu = -\mu_o \mathcal{J}_\nu. \tag{2.3-6}$$

Also, one can define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.3-7}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{iE_1}{c} \\ -B_3 & 0 & B_1 & -\frac{iE_2}{c} \\ B_2 & -B_1 & 0 & -\frac{iE_3}{c} \\ \frac{iE_1}{c} & \frac{iE_2}{c} & \frac{iE_3}{c} & 0 \end{pmatrix}$$

Then the Maxwell's equations in Axiom II would become

$$\partial_\nu F_{\mu\nu} = \mu_o \mathcal{J}_\nu \tag{2.3-8}$$

and

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = o \tag{2.3-9}$$

Now suppose that a second observer has $x'_\mu = (x'_i, ict')$

which may correspond to a simple rotation of the co-ordinate axes in 4-D Minkowski space. An extension of ideas form 3-D rotation implies that the transformation from x_μ to x'_μ should be orthogonal in the sense that $x_\mu x_\mu$ must remain invariant (note also that Axiom II implies the speed of light to have a constant value). Then one can, optionally, invoke Axiom III for special theory of relativity:

AXIOM III: All observers whose spacetime measurements are related by the orthogonal transformations in four dimensional Minkowski space are equivalent in formulating the "Laws of Physics". These laws must be formulated so as to avoid the initial conditions for different observers.

Thus the Axiom II, which expresses the laws of physics, must have the same form for different observers connected through the orthogonal transformation which is the Lorentz transformation. This last identification can be seen as follows. Suppose $x'_\mu = a_{\mu\nu} x_\nu$, then $x'_\mu x'_\mu = x_\mu x_\mu$ requires that

$$a_{\lambda\mu} a_{\lambda\nu} = \delta_{\mu\nu} \qquad (2.3\text{-}10)$$

and $$\det|a_{\mu\nu}| = 1 \quad . \qquad (2.3\text{-}11)$$

Now consider a rotation in Minkowski space. Rotations in x_1-x_2, x_2-x_3 and x_3-x_1 planes are the usual 3-D rotations. Rotation in x_3-x_4 plane by an angle Ψ is worth examining:

$$x'_1 = x_1$$
$$x'_2 = x_2$$
$$x'_3 = \mathrm{Cos}\,\psi\, x_3 + \mathrm{Sin}\,\psi\, x_4$$

$$x_4' = -\,\mathrm{Sin}\psi\, x_3 + \mathrm{Cos}\psi\, x_4 \;.$$

The space point $x_1' = o$, $x_2' = o$ and $x_3' = o$ (origin of primed co-ordinate system) implies

$$\mathrm{Cos}\psi\, x_3 = -\,\mathrm{Sin}\psi\, x_4$$

or

$$x_3 = -\tan\psi\,(ict),$$

provided $\psi \neq \pi/2$. As t increases, x_3 changes linearly with it so that the two space frames are moving relative to each other with speed $V = -\,ic\tan\psi$. It is customary to write this relation as

$$\tan\psi = iV/c \equiv i\beta$$

Hence one gets

$$a_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \qquad (2.3\text{-}12)$$

where $\gamma = 1/\sqrt{1-\beta^2}$. The rest of the story is as usual, but we next discuss the transormation properties of the electromagnetic quantities under Lorentz transformation.

3.1. Transformations of E-M Fields and Potentials

When two Lorentz frames move with relative velocity $\vec{v}$, one can easily construct $a_{\mu\nu}$ which assumes the simple form of equation (2.3-12) for $\vec{v}$ parallel to z-axes. Then the transformation equations for the electromagnetic quantities are as follows:

$$J'_\mu = a_{\mu\nu} J_\nu \tag{3.1-1}$$

$$A'_\mu = a_{\mu\nu} A_\nu \tag{3.1-2}$$

$$F'_{\mu\nu} = a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma} \tag{3.1-3}$$

The transformation properties of $\vec{E}$ and $\vec{B}$ come through $F_{\mu\nu}$. (Homework:) It can be shown that if both $\vec{E}$ and $\vec{B}$ are non-zero in one Lorentz frame there always exists another Lorentz frame in which (a) only electric field is non-zero or (b) only magnetic field is non-zero. But no Lorentz frames exist for transforming a purely electric field to a purely magnetic field.

Let us consider a simple but useful example of the transformation of the four potential A_μ. This is the example of potentials produced by a point charge e which is moving with velocity v along the z-axis (x_3-axis) with respect to the primed co- ordinate system. There is only the electrostatic potential in the (proper frame) coordinate system in which the particle is at rest at the Origin. Thus $A_\mu = (0,0,0,\frac{i}{4\pi\varepsilon_o} \frac{e}{cr_o})$ where r_o is the distance form of the point of observation of the charge in the unprimed frame. Clearly r_o should be written in a covariant form before transformation. Since the four velocity $u_\mu = (0,0,0,ic)$, one can write $-cr_o = u_\mu x_\mu$ (because

x_μ is on the light cone : $x_\mu x_\mu = 0$ so that $x_\mu = (\vec{r}_o, ir_o))$.

Thus

$$A_\mu = (0,0,0, \frac{+ie}{4\pi\varepsilon_o(-u_\mu x_\mu)})$$

and therefore

$$A'_\mu = \frac{e}{4\pi\varepsilon_o}(0,0, \frac{v}{c^2 s}, \frac{+i}{cs}) \qquad (3.1\text{-}4)$$

where

$$\begin{aligned} s &= -u'_\mu x'_\mu /(\gamma c) \\ &= r' - (\vec{r}' \cdot \vec{v})/c \end{aligned} \qquad (3.1\text{-}5)$$

in the general case.

Then the vector potential is

$$\vec{A}' = \frac{e\mu_o}{4\pi}\frac{\vec{v}}{s} \quad , \qquad (3.1\text{-}6)$$

and the scalar potential is

$$\phi = \frac{e}{4\pi\varepsilon_o}\frac{1}{s} \quad . \qquad (3.1\text{-}7)$$

These Lienard-Wiechert potentials are useful in radiation theory.

3.2 Lagrangian Density for Electromagnetic Field

The Lagrangian density should be written so as to yield the correct field equations. Consider the free field equation

$$\partial_\mu \partial_\mu A_\nu = 0 \quad , \qquad (3.2\text{-}1)$$

along with the condition

$$\partial_\mu A_\mu = 0 \quad . \qquad (3.2\text{-}2)$$

Now, equation (1.3-8) can be rewritten in this case in terms of Lagrangian density L as

$$\partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right] = 0 \; . \qquad (3.2\text{-}3)$$

L is a scalar and should be so chosen that equation (3.2-3) yields equation (3.2-1), subject to the Lorentz condition (3.2-2). The only possible scalars are $A_\mu A_\mu$, $\partial_\mu A_\nu \partial_\mu A_\nu$ and $\partial_\mu A_\nu \partial_\nu A_\mu$. It is easily seen that

$$L = \lambda_1 \partial_\mu A_\nu \partial_\nu A_\nu + \lambda_2 \partial_\mu A_\nu \partial_\mu A_\mu \qquad (3.2\text{-}4)$$

is a suitable choice. To determine λ_1 and λ_2 one can proceed as follows: Look at the free field equation

$$\partial_\mu F_{\mu\nu} = 0$$

which is without Lorentz condition, or

$$\partial_\mu \partial_\mu A_\nu - \partial_\mu \partial_\nu A_\mu = 0 \quad .$$

A comparison with equation (3.2-3) using equation (3.2-4) shows that $\lambda_2 = -\lambda_1$. Therefore $L = \lambda_1(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu)$. Finally, consider the nonzero J_μ case with Lorentz condition:

$$\partial_\mu F_{\mu\nu} = -\mu_o J_\nu$$

or
$$\partial_\mu \partial_\mu A_\nu = -\mu_o J_\nu \qquad (3.2\text{-}5)$$

One more scalar admissible in L is $J_\mu A_\mu$. The constant co-efficient for this term may be chosen to be unity because $J_\mu A_\mu$ already has the dimensions of energy. Then $L = \lambda_1 [\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu] + J_\mu A_\mu$ and the equation of motion is

$$2\lambda_1 \partial_\mu \partial_\mu A_\nu = J_\nu .$$

Comparison of this equation with equation (3.2-5) yields $\lambda_1 = -1/(2\mu_o) = -\varepsilon_o c^2/2$, so that the Lagrangian density is

$$L = -\frac{\varepsilon_o c^2}{2}(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) + \mathcal{J}_\mu A_\mu \qquad (3.2\text{-}6)$$

This is not unique because $(\partial_\mu A_\mu)$ or any funcion of $(\partial_\mu A_\mu)$ can be added to L without affecting the equation of motion. However, in Lorentz Gauge, $\partial_\mu A_\mu = 0$ and one would have liked to forget about it except for the fact that such terms need be retained in L for covariant quantization of e-m field as in the Gupta-Bleuler formalism.

Several equivalent forms of equation (3.2-6) are possible:

$$L = -\frac{\varepsilon_o c^2}{4} F_{\mu\nu} F_{\mu\nu} + J_\mu A_\mu \qquad (3.2\text{-}7)$$

$$L = (\frac{1}{2}\varepsilon_o E^2 - \frac{1}{2}\frac{1}{\mu_o} B^2) + J_\mu A_\mu \qquad (3.2\text{-}7)$$

This last form of equation (3.2-7) is interesting because the free field Lagrangian density is a difference of two terms whose mechanical analogues are kinetic and potential energies. The non unique feature of the Lagrangian density has led to many other forms of L, one of the more famous ones being the Fermi Lagrangian.

3.3 Noether's Theorem

In classical mechanics, the symmetry with respect to chang - ing or relabelling of generalized co-ordinates of a system leads to the conservation laws. Noether's theorem is the counterpart of this procedure for fields and states that:

"To every continuous co-ordinate transformation (group) which leaves the action integral invariant (and for which the transformation law of the field variable is specified), there corresponds a quantity (expressible in terms of field variables and their derivatives) which remains conserved."

There are other equivalent statements of this theorem in the literature. To prove this theorem, consider the invariance under the continuous transformation $x_\nu \to x'_\nu$. It is enough to take infinitesimal transformations

$$x_\nu \to x'_\nu = x_\nu + \delta x_\nu , \qquad (3.3\text{-}1)$$

whose repeated applications generate any finite transformation. We must also specify the transformation of field variables $\psi_\alpha(x_\nu)$ under the transformation of equation (3.3-1). Thus

$$\psi_\alpha(x_\nu) \to \psi'_\alpha(x'_\nu) = \psi_\alpha(x_\nu) + (\psi'_\alpha(x'_\nu) - \psi_\alpha(x_\nu))$$

$$= \psi_\alpha(x_\nu) + \psi'_\alpha(x'_\nu) - \psi_\alpha(x'_\nu) + \psi_\alpha(x'_\nu) - \psi_\alpha(x_\nu)$$

or $$\psi'_\alpha(x'_\nu) = \psi_\alpha(x_\nu) + \delta\psi_\alpha(x_\nu) + \partial_\mu \psi_\alpha(x_\nu)\, \delta x\mu \qquad (3.3\text{-}2)$$

One must specify the form change

$$\delta\psi_\alpha(x_\nu) = \psi'_\alpha(x'_\nu) - \psi_\alpha(x'_\nu), \qquad (3.3\text{-}3)$$

if there is any, with the transformation of equation (3.3-1). Now we demand invariance of the action integral under this transformation:

$$\Delta \int L d4x = 0 \qquad (3.3\text{-}4)$$

Note that this involves also the variation of the domain of integration. Therefore we write (3.3-4) as

$$\int_R L d4x = \int_{R'} L' d4x' \tag{3.3-5}$$

where R and R' denote the domains of integration. One can use the same domains if the Jacobian $J = \det(\frac{\partial x'_\mu}{\partial x_\nu})$ is introduced i.e. $d^4x' = Jd^4x$. For the infinitesimal transformation of equation (3.3-1), the 4x4 determinant up to first order in infinitesimals is

$$J = \det \begin{bmatrix} 1 + \partial_1(\delta x_1) \ldots\ldots\ldots & & & \\ & 1+\partial_2(\delta x_2) \ldots\ldots\ldots\ldots & & \\ & & 1+\partial_3(\delta x_3) \ldots\ldots\ldots\ldots & \\ & & & 1+\partial_4(\delta x_4) \ldots\ldots\ldots \end{bmatrix}$$

or $J \simeq 1+\partial_\mu(\delta x_\mu)$. (3.3-6)

Then equation (3.3-5) can be written as

$$\int \left[L'J-L\right] d4x = 0 \quad ,$$

or

$$\int \left[L'-L + L'\partial_\mu(\delta x_\mu)\right] d4x = 0 \quad . \tag{3.3-7}$$

But L'-L has contributions from $\delta\psi_\alpha$, $\delta(\partial_\mu\psi_\alpha)$ and δx_μ. Thus

$$\begin{aligned} L'-L &= \frac{\partial L}{\partial \psi_\alpha}\delta\psi_\alpha + \frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\delta(\delta_\mu\psi_\alpha) + (\partial_\mu L)\delta x_\mu \\ &= \frac{\partial L}{\partial \psi_\alpha}\delta\psi_\alpha + \frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\partial_\mu(\delta\psi_\alpha) + (\partial_\mu L)\delta x_\mu \\ &= \frac{\partial L}{\partial \psi_\alpha} - \partial_\mu\{\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\}\delta\psi_\alpha + \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\delta\psi_\alpha\right] + (\partial_\mu L)\delta x_\mu \end{aligned} \tag{3.3-8}$$

Substituting this in equation (3.3-7), the independent variation of $\delta\psi_\alpha$ implies the field equation

$$\partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\right] - \frac{\partial L}{\partial\psi_\alpha} = 0 \ . \tag{3.3-9}$$

Then equation (3.3.-8) yields

$$L'-L = \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\delta\psi_\alpha\right] + (\partial_\mu L)\ \delta x\mu. \tag{3.3-10}$$

Now

$$L'\partial_\mu(\delta x\mu) = L+\partial_\mu\{\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\delta\psi_\alpha\} + (\partial_\mu L)\delta x\mu\ \partial_\mu\ (\delta x\mu)$$

$$\simeq L\partial_\mu(\delta x_\mu)\ , \tag{3.3-11}$$

up to first order infinitesimals. Then equation (3.3-7) becomes

$$\int d^4x\partial_\mu\left[L\delta x\mu + \frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\ \delta\psi_\alpha\right] = 0 \tag{3.3-12}$$

To get the conservation laws from this equation, let us parametrize the transformation group by s <u>independent infinitesimal parameters</u> $w_1, w_2, \ldots w_s$.

Thus $\quad \delta x\mu = X^j_\mu\ w_j, \quad (j = 1,\ldots,s)\ ,$ $\hfill (3.3\text{-}13)$

so that $\quad \delta\psi_\alpha = S^j_\alpha\ w_j - \delta x_\nu\ \partial_\nu\ \psi_\alpha\ .$ $\hfill (3.3\text{-}14)$

Then equation (3.3-12) yields

$$\partial_\mu f^j_\mu = 0\ , \tag{3.3-15}$$

where

$$f^j_\mu = L\ X^j_\mu + \frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\ S^j_\alpha - \frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}\ X^j_\nu\ \partial_\nu\ \psi_\alpha\ . \tag{3.3-16}$$

Finally, since equation (3.3-15) is an equation of continuity, one can enclose a volume of space such that the fields vanish on its surface. Then the conserved quantities are integrals over this volume:

$$\Theta_j = \frac{1}{ic} \int f_4^j \, d^3x \tag{3.3-17}$$

Thus, the s quantities $\Theta_1, \ldots \Theta_s$ are constants in time. It must be emphasized that f_μ^j are arbitrary up to an additional quantity ϕ_μ^j which is the solution of $\partial_\mu \phi_\mu^j = 0$. This property is sometimes used to write f_μ^j in a suitable form.

3.4 Homogeneity of Space-time and Conservation of Energy-Momentum

If the field is not affected by infinitesimal shifts ε_μ of the spacetime origin in the Minkowski space, then

$$x'_\mu = x_\mu + \varepsilon_\mu \tag{3.4-1}$$

But $dx'_\mu = dx_\mu$ because ε_μ are constants and therefore the field equations do not change. Consequently, the only change in ψ_α comes through the derivative term in equation (3.3-14). Since there are four ε_μ, we identify

$$\begin{aligned} \delta x_\mu &= \varepsilon_\mu \, , \\ X_\nu^\mu &= \delta_{\nu\mu} \, , \\ \text{and} \qquad S_\alpha^\mu &= 0. \end{aligned} \tag{3.4-2}$$

The conserved quantities are volume integrals of f_4^μ which are the conserved densities for the fields. Under the transformation of equation (3.4-1), one gets

$$T_{\mu\nu} \equiv f_\mu^\nu = L\delta_{\mu\nu} - \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} \partial_\nu \psi_\alpha \tag{3.4-3}$$

which is called the canonical energy-momentum tensor for the field. The conserved quantities are

$$P_\mu = \frac{1}{ic} \int T_{4\mu} d^3x \ , \tag{3.4-4}$$

which constitute the energy-momentum four-vector for the field. Usually $T_{\mu\nu}$ so derived is not symmetric but adding an extra term whose four-divergence is equal to zero one can make $T_{\mu\nu}$ symmetric.

3.5 Isotropy of Space-time and Conservation of Angular Momentum

The isotropy of space-time would imply that rotations in Minkowski space (Lorentz transformations) do not affect the action integral for the field. Such rotations also contain the three dimensional rotations and hence lead to conservation of angular momentum. The infinitesimal rotations can be written as

$$x'_\mu = x_\mu + \varepsilon_{\mu\nu} x_\nu \tag{3.5-1}$$

We recall from section 2.3 that for this case

$$a_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}$$

But equation (2.3-10) gives

$$\delta_{\mu\nu} = a_{\lambda\mu} a_{\lambda\mu} = (\delta_{\lambda\mu}+\varepsilon_{\lambda\mu})(\delta_{\lambda\mu}+\varepsilon_{\lambda\mu})$$

$$= \delta_{\mu\nu}+\varepsilon_{\nu\mu}+\varepsilon_{\mu\nu}+O(\varepsilon^2) \ ,$$

whence $$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \ . \tag{3.5-2}$$

This antisymmetry implies that <u>only 6 out</u> of the 16 quantities $\varepsilon_{\mu\nu}$ are <u>independent</u>. Therefore we should expect six conserved quantities. It is important to remember that the parameter index j now has two indices $\nu\lambda$ which are not all independent in view of the antisymmetry of $\varepsilon_{\nu\lambda}$. Then one can proceed as follows. First, we have

$$\delta x_\mu = \varepsilon_{\mu\nu} x_\nu \tag{3.5-4}$$

Then the specification of $\delta\psi_\alpha$ can be done by defining

$$\psi'_\alpha(x') = \tau_{\alpha\beta}\ \psi_\beta(x), \tag{3.5-5}$$

so that $\delta\psi_\alpha = (\tau_{\alpha\beta} - \delta_{\alpha\beta})\ \psi_\beta(x) - \partial_\nu \psi_\alpha(x)\, \delta x_\nu$

But $\tau_{\alpha\beta}$ is a function of $\varepsilon_{\mu\nu}$ and may be expanded as

$$\tau_{\alpha\beta}(\varepsilon_{\mu\nu}) = \tau_{\alpha\beta}(o) + \left.\frac{\partial \tau_{\alpha\beta}}{\partial \varepsilon_{\mu\nu}}\right|_{\varepsilon_{\mu\nu}=0} \varepsilon_{\mu\nu} + O(\varepsilon^2) \ldots.$$

$$= \delta_{\alpha\beta} + t_{\alpha\beta\mu\nu}\varepsilon_{\mu\nu} + O(\varepsilon^2) \ldots.,$$

where the first derivative has been called $t_{\alpha\beta\mu\nu}$. Thus

$$\delta\psi_\alpha = t_{\alpha\beta\mu\nu}\ \varepsilon_{\mu\nu}\ \psi_\beta - \partial_\nu \psi_\alpha\, \delta x_\nu\ . \tag{3.5-6}$$

It follows from equation (3.3-12) that $\partial_\mu f_\mu = 0$ where

$$\begin{aligned} f_\mu &= L\delta x\mu - \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)}\ \partial_\nu \psi_\alpha\ \delta x_\nu + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)}\ \delta\ \psi_\alpha \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} \partial_\nu \psi_\alpha\ \ \delta x_\nu + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)}\ t_{\alpha\beta\nu\lambda}\ \varepsilon_{\nu\lambda}\ \psi_\beta \\ &= T_{\mu\nu}\ \delta x_\nu + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)}\ t_{\alpha\beta\nu\lambda}\ \psi_\beta\ \varepsilon_{\nu\lambda} \end{aligned} \tag{3.5-7}$$

Here, equation (3.4-3) has been used to identify the energy momentum tensor $T_{\mu\nu}$. Further, we can write using equation (3.5-4)

$$f_\mu = \left[T_{\mu\nu} x_\lambda + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} t_{\alpha\beta\nu\eta} \psi_\beta\right] \varepsilon_{\nu\lambda}$$

but since all $\varepsilon_{\nu\lambda}$ are not independent, one should symmetrize by writing

$$\begin{aligned} 2f_\mu &= \left[T_{\mu\nu} x\lambda + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} t_{\alpha\beta\nu\lambda} \psi_\beta\right] \varepsilon\nu\lambda \\ &+ \left[T_{\mu\lambda} x_\nu + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} t_{\alpha\beta\lambda\nu} \psi_\beta\right] \varepsilon\lambda\nu \\ &= \left[(T_{\mu\nu} x\lambda - T_{\mu\lambda} x_\nu) + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} s_{\alpha\beta\nu\lambda} \psi_\beta\right] \varepsilon_{\nu\lambda}, \end{aligned}$$

where now the $\varepsilon_{\nu\lambda}$ may be treated as independent parameters and

$$s_{\alpha\beta\nu\lambda} = (t_{\alpha\beta\nu\lambda} - t_{\alpha\beta\nu\lambda}) \, . \qquad (3.5\text{-}8)$$

Thus, the equation $\partial_\mu f_\mu = 0$ may now be written as

$$\partial_\mu m_{\mu\nu\lambda} = 0 \, , \qquad (3.5\text{-}9)$$

where

$$m_{\mu\nu\lambda} = T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu + \frac{\partial L}{\partial(\partial_\mu \psi_\alpha)} s_{\alpha\beta\nu\lambda} \psi_\beta \qquad (3.5\text{-}10)$$

This tensor is antisymmetric in ν and λ. The six conserved quantities have the densities

$$\begin{aligned} M_{\mu\nu} &= -M_{\nu\mu} \equiv \frac{1}{ic} m_{4\mu\nu} = \frac{1}{ic} (T_{4\mu} x_\nu - T_{4\nu} x_\mu) \\ &+ \frac{1}{ic} \frac{\partial L}{\partial(\partial_4 \psi_\alpha)} s_{\alpha\beta\mu\nu} \psi_\beta \end{aligned} \qquad (3.5\text{-}11)$$

It is not difficult to realize that the space part of the first term will lead to the usual orbital angular momentum density. The space part of $\frac{1}{ic} T_{4\mu}$ are momentum density components of the field. But the space part of the second term (which in general may not vanish) has the same dimensions as angular momentum density although it has nothing to do with the orbital motion. However it is the sum of the two terms that are conserved. We may call the second term as 'spin' angular momentum density. It must be emphasized that this is an intrinsic property of the field which cannot be represented like an orbital angular momentum. Moreover, this has appeared in a purely classical formalism. A scalar field does not change under Lorentz transformation and hence has zero 'spin'. Other fields <u>may</u> have 'spin'. In fact, application of quantum mechanics can show that a tensor field of rank n has spin $n\hbar$. The vector field of electrodynamics has spin 1 in quantum theory and will have some 'spin' in the classical theory presented above.

4.1 Energy-Momentum of Electromagnetic Field

From equation (3.3-12) of Noether's theorem and the transformation (3.4-1) one gets $\partial_\mu T_{\mu\nu} = 0$ where for e-m field (with $\psi_\alpha = A_\alpha$)

$$T_{\mu\nu} = L\,\delta_{\mu\nu} - \frac{\partial L}{\partial(\partial_\mu A_\alpha)}\,\partial_\nu A_\alpha \tag{4.1-1}$$

But the Lagrangian density for e-m field is given by (3.2-6) whose free field part is

$$\begin{aligned} L &= -\frac{\epsilon_0 c^2}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \\ &= -\frac{\epsilon_0 c^2}{4} F_{\mu\nu} F_{\mu\nu} \quad . \end{aligned} \tag{4.1-2}$$

Then
$$\frac{\partial L}{\partial(\partial_\mu A_\alpha)} = -\epsilon_0 c^2 F_{\mu\nu} \quad , \tag{4.1-3}$$

and
$$T_{\mu\nu} = \epsilon_0 c^2 \left[F_{\mu\alpha}\,\partial_\nu A_\alpha - \tfrac{1}{4} F_{\alpha\beta} F_{\alpha\beta}\,\delta_{\mu\nu}\right] .$$

This may be symmetrized by adding a term

$$t_{\mu\nu} = \epsilon_0 c^2 \left[-F_{\mu\alpha}\,\partial_\alpha A_\nu\right] \quad ,$$

because $\partial_\mu t_{\mu\nu} = 0$. (Homework: prove this).

Then

$$T_{\mu\nu} = \epsilon_0 c^2 \left[F_{\mu\alpha} F_{\nu\alpha} - \tfrac{1}{4} F_{\alpha\beta} F_{\alpha\beta}\,\delta_{\mu\nu}\right] . \tag{4.1-4}$$

Let us write this tensor as

$$T_{\mu\nu} = \begin{pmatrix} -T^M_{11} & -T^M_{12} & -T^M_{13} & \frac{i}{c} N_1 \\ -T^M_{21} & -T^M_{22} & -T^M_{23} & \frac{i}{c} N_2 \\ -T^M_{31} & -T^M_{32} & -T^M_{33} & \frac{i}{c} N_3 \\ \frac{i}{c} N_1 & \frac{i}{c} N_2 & \frac{i}{c} N_3 & -H \end{pmatrix} \tag{4.1-5}$$

Then

$$H = -\epsilon_0 c^2 \left[F_{4\alpha} F_{4\alpha} - \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} \right]$$
$$= \epsilon_0 c^2 \left[\frac{1}{4} \left(2B^2 - 2E^2/c^2 \right) + E^2/c^2 \right]$$
$$= \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \quad , \tag{4.1-6}$$

where equation (2.3-7) for $F_{\mu\nu}$ has been used. H is the Hamiltonian density. It is the energy density of the field. Let us evaluate the other quantities in equation (4.1-5)

$$N_1 = -ic\, T_{41}$$
$$= -ic\, \epsilon_0 c^2 F_{4\alpha} F_{1\alpha} = \epsilon_0 c^2 \left(E_2 B_3 - E_3 B_2 \right)$$

Thus
$$\vec{N} = \left(\vec{E} \times \vec{B} \right) / \mu_0 \quad , \tag{4.1-7}$$

is the Poynting vector for the field.

Finally

$$T^M_{\ell\ell} = -T_{\ell\ell} = -\epsilon_0 c^2 \left[F_{\ell\alpha} F_{\ell\alpha} - \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} \right]$$
$$= -\epsilon_0 c^2 F_{\ell\alpha} F_{\ell\alpha} - \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

Thus
$$T^M_{11} = \epsilon_0 E_1^2 - \left(B_2^2 + B_3^2 \right)/\mu_0 - \frac{1}{2} \epsilon_0 E^2 + B^2/(2\mu_0)$$
$$= \epsilon_0 E_1^2 + B_1^2/\mu_0 - \left(\frac{1}{2} \epsilon_0 E^2 + B^2/(2\mu_0) \right)$$

Also for $k \neq \ell$

$$T^M_{k\ell} = -T_{k\ell} = -\epsilon_0 c^2 F_{k\alpha} F_{\ell\alpha}$$
$$= \epsilon_0 E_k E_\ell + B_k B_\ell / \mu_0 \quad .$$

Then one gets in general

$$T^M_{k\ell} = \epsilon_0 E_k E_\ell + B_k B_\ell / \mu_0 - \delta_{k\ell} \left(\frac{1}{2} \epsilon_0 E^2 + B^2/(2\mu_0) \right) . \tag{4.1-8}$$

This is the Maxwell's stress tensor. The usual interpretations of

H, $\vec{N}$ and T^M_{kl} follow from $\partial_\mu T_{\mu\nu} = 0$. Thus $\partial_\mu T_{\mu 4} = 0$ gives

$$\nabla \cdot \vec{N} + \frac{\partial}{\partial t} H = 0 \; ; \tag{4.1-9}$$

whence
$$\frac{\partial}{\partial t} \int_V H \, d^3x = - \oint_S \vec{N} \cdot d\vec{s} \, , \tag{4.1-10}$$

which describes the flow of energy from a volume V enclosed by the surface S.

Therefore, the Poynting vector gives the rate of flow of energy per unit area. However, one must be careful that equation (4.1-10) originates from equation (4.1-9) so that there is no energy flow for static fields although $\vec{N}/c^2$ is the momentum density for the field (in analogy with the definition in mechanics).

In presence of currents J_μ, the full L of equation (3.2-6) gives (in view of the field equation (3.2-5))

$$\partial_\mu T_{\mu\nu} = - F_{\nu\alpha} J_\alpha \quad . \tag{4.1-11}$$

Then,
$$\nabla \cdot \vec{N} + \frac{\partial H}{\partial t} = - \vec{E} \cdot \vec{j} \quad . \tag{4.1-12}$$

gives the dissipation of energy due to currents. Also one gets

$$\partial_\mu T_{\mu k} = - J_\alpha F_{k\alpha}$$

or
$$\partial_l T^M_{lk} - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})_k = [\rho \vec{E} + \vec{j} \times \vec{B}]_k \; . \tag{4.1-13}$$

The right hand side of this equation is the volume force on charge distribution (Lorentz force per unit volume). This equals the divergence of Maxwell stress tensor and a time rate of change of field momentum. Thus, T^M_{lk} represent a kind of stress in the field.

The homogeneity of space time thus leads to conservation of 4 quantities: The energy and momentum components of the field.

4.2 Angular Momentum of the Electromagnetic Field

Consider the Lorentz transformations $x'_\mu = a_{\mu\nu} x_\nu$ whose infinitesimal form will give $a_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}$ with the six parameter antisymmetric inifinitesimal tensor $\varepsilon_{\mu\nu}$ described in equation (3.5-2). We also have

$$A'_\mu (x') = a_{\mu\nu} A_\nu (x) \qquad (4.2\text{-}1)$$

so that $\quad \delta A_\alpha = \epsilon_{\alpha\nu} A_\nu - (\partial_\mu A_\alpha) \delta x_\mu \quad .$

Then, from equation (3.3-12)

$$\partial_\mu f_\mu = 0 \quad , \qquad (4.2\text{-}2)$$

where

$$\begin{aligned} f_\mu &= L\, \delta x_\mu + \frac{\partial L}{\partial(\partial_\mu A_\alpha)} \delta A_\alpha \\ &= L\, \epsilon_{\mu\nu} x_\nu + \frac{\partial L}{\partial(\partial_\mu A_\alpha)} \epsilon_{\alpha\nu} A_\nu - \frac{\partial L}{\partial(\partial_\mu A_\lambda)} \partial_\alpha A_\lambda \epsilon_{\alpha\nu} x_\nu \\ &= \left[L\, \delta_{\mu\alpha} - \frac{\partial L}{\partial(\partial_\mu A_\lambda)} \partial_\alpha A_\lambda \right] \epsilon_{\alpha\nu} x_\nu + \frac{\partial L}{\partial(\partial_\mu A_\alpha)} \epsilon_{\alpha\nu} A_\nu \\ &= \left(T_{\mu\alpha} x_\nu - \epsilon_0 c^2 F_{\mu\alpha} A_\nu \right) \epsilon_{\alpha\nu} \quad . \end{aligned} \qquad (4.2\text{-}3)$$

Here we have used the energy-momentum tensor of equation (4.1-1), the result of equation (4.1-3) and free field part of L in equation (3.2-6). The summation over α and ν contain $\varepsilon_{\alpha\nu}$ which are not at all independent. Therefore one should properly antisymmetrize the co-efficients by writing

$$f_\mu = \frac{1}{2}\left[(T_{\mu\alpha}x_\nu - \epsilon_0 c^2 F_{\mu\alpha}A_\nu)\epsilon_{\alpha\nu} + (T_{\mu\nu}x_\alpha - \epsilon_0 c^2 F_{\mu\nu}A_\alpha)\epsilon_{\nu\alpha}\right]$$

$$= \frac{1}{2} m_{\mu\alpha\nu}\epsilon_{\alpha\nu} \quad ,$$

where

$$m_{\mu\alpha\nu} = (T_{\mu\alpha}x_\nu - T_{\mu\nu}x_\alpha) - \epsilon_0 c^2(F_{\mu\alpha}A_\nu - F_{\mu\nu}A_\alpha) \ .$$

The six conserved densities derived from $\partial_\mu m_{\mu\alpha\nu} = 0$, are

$$M_{\mu\nu} = -M_{\nu\mu} = \frac{i}{c} m_{4\mu\nu}$$

$$= \frac{i}{c}(T_{4\mu}x_\nu - T_{4\nu}x_\mu) - i\epsilon_0 c(F_{4\mu}A_\nu - F_{4\nu}A_\mu) \quad (4.2\text{-}4)$$

The orbital angular momentum part is the first term and the second term contains the spin. Consider

$$j_3 = M_{12} = \frac{i}{c}(T_{41}x_2 - T_{42}x_1) - i\epsilon_0 c(F_{41}A_2 - F_{42}A_1)$$

$$= -(N_1x_2 - N_2x_1)/c^2 + \epsilon_0(E_1A_2 - E_2A_1)$$

$$= -(\vec{N}\times\vec{r})_3/c^2 + \epsilon_0(\vec{E}\times\vec{A})_3 \quad ,$$

or

$$\vec{j} = \vec{\ell} + \vec{\sigma} \, , \quad (4.2\text{-}5)$$

where

$$\vec{\ell} = \vec{r}\times(\vec{N}/c^2) = \epsilon_0\vec{r}\times(\vec{E}\times\vec{B}) \, , \quad (4.2\text{-}6)$$

is the orbital angular momentum density (recollect that $\vec{N}/c^2 = \epsilon_0(\vec{E}\times\vec{B})$ is the momentum density). Also

$$\vec{\sigma} = \epsilon_0(\vec{E}\times\vec{A}) \quad (4.2\text{-}7)$$

is the spin angular momentum density. These two cannot in general be separated. For the special case of a polarized plane wave moving along x_3 (+ve) direction, with

$$A_1 = \left(\frac{a}{\omega}\right)\sin(kx_3 - \omega t) \quad , \quad A_2 = \left(\frac{b}{\omega}\right)\sin(kx_3 - \omega t + \delta) \, ,$$

the fields are $E_1 = a\cos(kx_3 - \omega t)$, $B_1 = (-b/c)\cos(kx_3 - \omega t + \delta)$, $E_2 = b\cos(kx_3 - \omega t + \delta)$ and $B_2 = (a/c)\cos(kx_3 - \omega t)$.

The $\vec{I}$ lies in (x_1, x_2) plane whose integration yields zero orbital angular momentum. The spin angular momentum is nonvanishing. In fact

$$\sigma_3 = \epsilon_0 (E_1 A_2 - E_2 A_1) \quad ,$$

$$\text{or} \qquad \sigma_3 = \epsilon_0 (ab/\omega) \sin\delta \tag{4.2-8}$$

Now the time average of the total energy of the plane wave is

$$\begin{aligned} \bar{W} &= \int \frac{\epsilon_0 c^2}{2} \left(\overline{B^2 + E^2/c^2} \right) d^3x \\ &= (\epsilon_0/2) \int (a^2 + b^2)\, d^3x \end{aligned}$$

Thus, the ratio of total spin to the total energy for the polarized plane wave is

$$\frac{\vec{\Sigma}}{\bar{W}} = \hat{k} \left(\frac{2ab}{a^2+b^2} \right) \frac{\sin\delta}{\omega} \tag{4.2-9}$$

This formula of Abraham and Sommerfeld yields zero spin for plane polarized light ($\delta = 0$). For right-circular polarization, $a = b$ and $\delta = \frac{\pi}{2}$ so that $\vec{\Sigma} = \hat{k}(\bar{W}/\omega)$, which when combined with Planck's $\bar{W} = \hbar\omega$ yields $\vec{\Sigma} = \hbar\hat{k}$.

So far, we have considered only the three conserved densities $\vec{j}$ which correspond to the rotaion of space co-ordinates. The 'pure' Lorentz transformations are rotations involving the time axis. Now consider the quantity

$$j_1' = M_{14} = (T_{41} x_4 - T_{44} x_1) - \epsilon_0 c^2 (F_{41} A_4 - F_{44} A_1) .$$

The last term here may be dropped if the symmetrized $T_{\mu\nu}$ of equation (4.1-5) is used. This is due to the addition of $t_{\mu\nu}$ which can cast this second term into divergence whose integration often vanishes. Then one gets for the free field,

$$\frac{d}{dt}\int \vec{j}' d^3x = \frac{d}{dt}\int H\vec{r}\, d^3x - \frac{d}{dt}\int t\vec{N}\, d^3x = 0 \; . \qquad (4.2\text{-}10)$$

Now define 'position of center of energy'

$$\vec{R} = \left(\int H\vec{r}\, d^3x\right)/\left(\int H\, d^3x\right) ,$$

and call

$$M_{field} = \frac{1}{c^2}\int H\, d^3x \quad \text{and} \quad \vec{P}_{field} = \frac{1}{c^2}\int \vec{N}\, d^3x .$$

Then

$$M_{field}\frac{d\vec{R}}{dt} = \frac{1}{c^2}\int \vec{N}\, d^3x = \int (\vec{N}/c^2)\, d^3x \quad , \qquad (4.2\text{-}11)$$

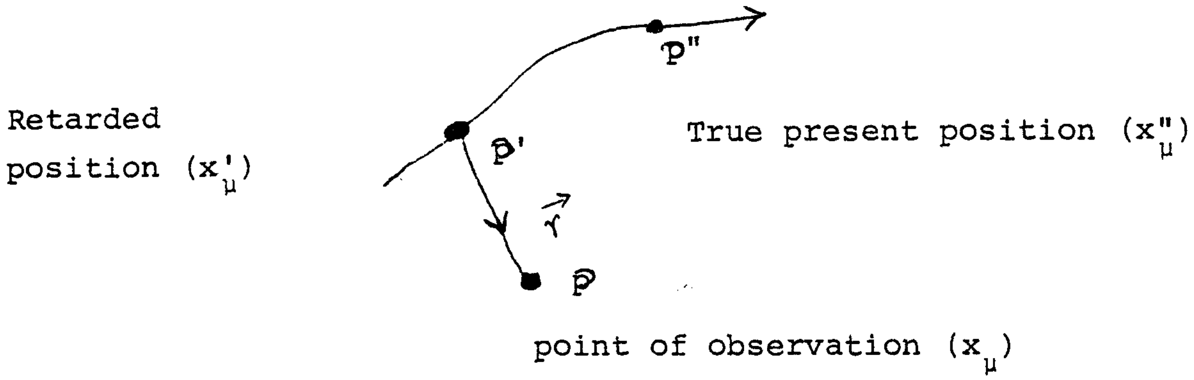
using equation (4.2-10). Hence

$$M_{field}\frac{d\vec{R}}{dt} = \vec{P}_{field} \; . \qquad (4.2\text{-}12)$$

This is the center of mass theorem for free electromagnetic field confined in space.. It states that the velocity of the center of mass is a constant and equals the total field momentum divided by the total field mass. We conclude this section by pointing out that the six quantities $M_{\mu\nu}$ together with the four quantities $T_{4\nu}$ are generators of full Lorentz group. But the Maxwell's equations are invariant under a larger group called Conformal group which has 15-parameters.

4.3 Fields due to an Accelerated Electron

I would like to confine the rest of the lectures to a discussion of electron whose charge is -e. Here we consider the radiation from an accelerated electron.



We are supposed to know the point of observation x_μ and the retarded position x'_μ. We also know $u_k = dx'_k/dt'$ and $\dot{u}_k = d^2x'_k/dt'^2$ of the electron at its retarded position P'. At the time t of observation, the electron has moved to some other position P". If P'P is defined as $\vec{r}$, then

$$r(x_k, x'_k(t')) = (x_k - x'_k)(x_k - x'_k) = c(t - t'), \tag{4.3-1}$$

which is the retardation condition. The Lienard-Wiechert potentials are given by equations (3.1-6) and (3.1-7) with

$$s = r - (\vec{r} \cdot \vec{u})/c \tag{4.3-2}$$

We wish to calculate the corresponding fields. Consider equation (2.2-2) for $\vec{E}$. We get

$$-\frac{4\pi\epsilon_0}{e}\vec{E} = \frac{1}{s^2}\nabla s - \frac{\partial}{\partial t}\left(\frac{\vec{u}}{sc^2}\right). \tag{4.3-3}$$

One must evaluate the derivatives with respect to x'_μ. For this purpose, we note that

$$\frac{\partial r}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right) = \frac{\partial r}{\partial t'}\frac{\partial t'}{\partial t} = -\frac{(x_k - x'_k)}{r}\frac{\partial x'_k}{\partial t'}\left(\frac{\partial t'}{\partial t}\right) = -\frac{\vec{r}\cdot\vec{u}}{r}\frac{\partial t'}{\partial t}$$

or

$$\frac{\partial t'}{\partial t} = 1/\left(1 - \frac{\vec{r}\cdot\vec{u}}{rc}\right) = \frac{r}{s} \tag{4.3-4}$$

Also

$$\partial_1 r = -c\,\partial_1 t' = \partial_1 r\Big|_{t'} + \frac{\partial r}{\partial t'}\partial_1 t'$$
$$= \frac{x_1}{r} - \frac{\vec{r}\cdot\vec{u}}{r}\partial_1 t'$$

or

$$\partial_1 t' = -x_1/(cs)\,. \qquad (4.3\text{-}5)$$

Finally , $\partial_1 s = \partial_1 s\Big|_{t'} + \frac{\partial s}{\partial t'}\partial_1 t'$

$$= \left(\frac{x_1}{r} - \frac{u_1}{c}\right) - \frac{x_1}{cs}\frac{\partial s}{\partial t'}\ . \qquad (4.3\text{-}6)$$

Then equation (4.3-3) yields

$$-\frac{4\pi\epsilon_0}{e}\vec{E} = \frac{1}{s^2}\left(\frac{\vec{r}}{r} - \frac{\vec{u}}{c}\right) - \frac{1}{s^2}\frac{\vec{r}}{cs}\frac{\partial s}{\partial t'} - \frac{r}{s}\frac{\partial}{\partial t'}\left(\frac{\vec{u}}{c^2 s}\right),$$

which can be finally written as

$$-\frac{4\pi\epsilon_0}{e}\vec{E} = \frac{1}{s^3}\left(\vec{r} - \frac{r\vec{u}}{c}\right)\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2 s^3}\left\{\vec{r}\times\left[\left(\vec{r} - \frac{r\vec{u}}{c}\right)\times\dot{\vec{u}}\right]\right\}, \qquad (4.3\text{-}7)$$

and it can also be shown that

$$\vec{B} = \frac{\vec{r}\times\vec{E}}{rc}\,. \qquad (4.3\text{-}8)$$

The first part of the field is independent of $\dot{u}$ and varies as $1/r^2$ at large distances. This is the induction field which does not contribute to the flow of energy. The second term containing $\dot{u}$ and varying as $1/_r$ represents the radiation field.

5.1 Radiation from an Accelerated Electron: Damping Force:

The usual treatment of the radiation from an accelerated electron consists of taking the radiation fields of equation (4.3-7) and (4.3-8) i.e.

$$-\frac{4\pi\epsilon_0}{e}\vec{E}_{rad} = \frac{1}{c^2 s^3}\left\{\vec{r}\times\left[\left(\vec{r}-\frac{r\vec{u}}{c}\right)\times\dot{\vec{u}}\right]\right\}, \tag{5.1-1}$$

and

$$\vec{B}_{rad} = \frac{\vec{r}\times\vec{E}_{rad}}{rc}. \tag{5.1-2}$$

Then the Poynting vector will be

$$\begin{aligned}\vec{N}_{rad} &= \epsilon_0 c^2\, \vec{E}_{rad}\times\vec{B}_{rad} = \frac{\epsilon_0 c}{r}\,\vec{E}_{rad}\times\left[\vec{r}\times\vec{E}_{rad}\right]\\ &= \frac{\epsilon_0 c}{r}\,\vec{r}\,E^2_{rad}\\ &= \frac{e^2}{16\pi^2\epsilon_0 c^3}\,\frac{\vec{r}}{r s^6}\left[(\vec{r}\cdot\dot{\vec{u}})\left(\vec{r}-\frac{r\vec{u}}{c}\right)-\left(r^2-r\,\frac{\vec{r}\cdot\vec{u}}{c}\right)\dot{\vec{u}}\right]^2.\end{aligned} \tag{5.1-3}$$

For $(u/_c \ll 1)$, one gets

$$\vec{N}_{rad} = \frac{e^2}{16\pi^2\epsilon_0 c^3}\,\frac{\vec{r}}{r^7}\left[(\vec{r}\cdot\dot{\vec{u}})\,\vec{r}-r^2\dot{\vec{u}}\right]^2. \tag{5.1-4}$$

Consider a spherical surface of large radius r. The energy flowing out per unit time is

$$-\frac{\partial E}{\partial t} = \int \vec{N}_{rad}\cdot d\vec{S} = \frac{e^2}{6\pi\epsilon_0 c^3}(\dot{u})^2 \tag{5.1-5}$$

This famous formula of Larmor (1897) has given results in excellent agreement with experiments and played a very important role in electrodynamics. Radiations from accelerated charges in many cases including those in particle accelerators have been found to agree with equations (5.1-4) and (5.1-5). But these very formulae have been the source of many difficulties in classical electrodynamics which have been overshadowed by quantum ideas.

Since the electron gives out radiated energy, it must encounter a damping force. The external force that produces acceleration of the electron must also supply both the energy and momentum required by the change in fields. The changes in the fields also act back on the electron itself and produce an extra inertia represented by a mass m_{el}. For $u \ll c$, the vector potential changes as $\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 e}{4\pi r}\frac{\partial \vec{u}}{\partial t}$, whence an effective electrical field acts on the electron with a force $-e\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 e^2}{4\pi r}\dot{\vec{u}}$. Then $m_{el} \sim \left(\frac{\mu_0 e^2}{4\pi r_0}\right)$ which is infinite for a point electron but assuming the classical electron radius

$$r_0 = \frac{e^2}{4\pi \epsilon_0 m c^2} \tag{5.1-6}$$

one gets $m_{el} \sim m$. The reaction force will also have a part $\vec{F}_1$ responsible for the energy loss given by Larmor formula, i.e.

$$\vec{F}_1 \cdot \vec{u} + \frac{e^2 (\dot{\vec{u}})^2}{6\pi\epsilon_0 c^3} = 0 \tag{5.1-7}$$

This equation represents a serious difficulty because if $\vec{u}$ and $\dot{\vec{u}}$ are uncorrelated; there is no solution valid for all time. We can resort to the ususal 'trick' by saying that a solution representing averages over a "sufficiently" long period of time should be useful. Then

$$\int_{t_1}^{t_2} (\vec{F}_1 \cdot \vec{u})\, dt + \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_0 c^3} (\dot{u})^2\, dt = 0$$

or

$$\int_{t_1}^{t_2} \left[\vec{F}_1 - \frac{e^2 \ddot{\vec{u}}}{6\pi\epsilon_0 c^3}\right] \cdot \vec{u}\, dt + \left[\frac{e^2 \vec{u} \cdot \dot{\vec{u}}}{6\pi\epsilon_0 c^3}\right]_{t_1}^{t_2} = 0 \tag{5.1-8}$$

The last term represents a 'fluctuation' over the average energy balance and this fluctuation will be stored in the induction field. For periodic motions or accleration limited in time, one gets for the average energy conservation

$$\vec{F}_1 = \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\vec{u}} \tag{5.1-9}$$

Then the total damping or radiation reaction force will be

$$\vec{F}_r = -m_{el}\dot{\vec{u}} + \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\vec{u}} \quad , \tag{5.1-10}$$

and the equation of motion of the electron under an external force $\vec{F}$ is

$$\vec{F} + \frac{e^2\ \ddot{\vec{u}}}{6\pi\epsilon_0 c^3} = (m + m_{el})\dot{\vec{u}} \quad , \tag{5.1-11}$$

where m_{el} represents a mass renormalization.

5.2 Introduction of the Planck's constant $\hbar$:

As a small deviation from the above theme of radiation reaction, let us note that all the energy expressions will always contain e^2 factors. This is because $\vec{E}$ is proportional to e and energy expressions are ultimately quadratic in fields. In fact the factor that always appears is $e^2/(4\pi\epsilon_0 c)$. This factor has dimensions of angular momentum or energy multiplied by time and has a value of about ($1/137$) times the Planck's constant $\hbar$. In fact

$$\frac{e^2}{4\pi\epsilon_0 c} = \alpha\hbar \quad , \tag{5.2-1}$$

where the dimensionless constant α has the value

$$\alpha = 1/(137.03) \tag{5.2-2}$$

This quantity is also well known in atomic physics as the fine structure constant. There can be no dispute when we treat α as a purely dimensionless numerical constant. But it is certainly unusual to say that $\hbar$ is nothing new in radiation theory: it has always existed in the form of $\frac{e^2}{4\pi\epsilon_0 c}$ times a numerical factor. In fact, some people who regard $\hbar$ as a pure 'quantum' object may feel disturbed to see $\hbar$ in a classical radiation formula. But didn't Max Planck introduce $\hbar$ for black body radiation, which is a problem of electromagnetic radiation?

This is not to say that $\hbar$ is not an independent constant. In fact, if $\hbar$ is not treated as fundamental, then α has to be fundamental and equation (5.2-2) can be regarded as its experimental value. I am of the opinion that there is no harm in viewing α as fundamental at the present time and may be the future theory can relate it to something else. The advantage is that α is a numerical constant and it might even be a scale factor neglected somewhere in the theory. All these are speculations except that <u>there is nothing wrong in introducing $\hbar$ in radiation theory</u> through equation (5.2-1). Then we rewrite equations (5.1-5), (5.1-6), (5.1-7) and (5.1-10) as

$$-\frac{\partial E}{\partial t} = \frac{2}{3}\frac{\alpha\hbar}{m^2c^2}(\dot{p})^2 \quad , \tag{5.2-3}$$

$$r_0 = \frac{\alpha\hbar}{mc} \quad , \tag{5.2-4}$$

$$\vec{F}_1\cdot\vec{u} + \frac{2}{3}\frac{\alpha\hbar}{mc^2}(\dot{p})^2 = 0 \quad , \tag{5.2-5}$$

and

$$\vec{F}_r = -\left(\frac{m_{el}}{m}\right)\dot{\vec{p}} + \frac{2}{3}\frac{\alpha\hbar}{mc^2}\ddot{\vec{p}} \tag{5.2-6}$$

Here we have used $\vec{p} = m\vec{u}$ with the constant bare mass m for the electron. One can introduce $\hbar$ at many other places where (e^2/c) appears.

An interesting quantity is the time τ defined as

$$\tau = \frac{r_0}{c} = \frac{\alpha \hbar}{mc^2} \qquad (5.2\text{-}7)$$

which will be used later on. For electron, $\tau \sim 10^{-23}$ second.

5.3 Abraham-Lorentz Derivation of Radiation Reaction

The radiation reaction force represents the interaction of the electron with its own field. One way to calculate this force is to consider the field due to one part of the electron acting on the other parts of it. Here we outline such a derivation of $\vec{F}_r$, omitting the details of the mathematical steps (see any book like Jackson for details). Assume :

1) a rigid (no Lorentz Contraction) spherical charge distribution for electron with radius a,

2) $u \ll c$, $\dot{u} \ll c^2/a$, $\ddot{u} \ll \dot{u}c/a$ etc so that $\vec{u}$ and its time derivatives change negligibly over a time τ (equation (5.2-7) ,

3) that the part de' produces fields that act on another part de whose rest frame is used for calculation, and

4) that the fields can be derived from retarded Lienard-Wiechert potentials.

Then

$$-4\pi\epsilon_0 d\vec{E}(t) = \frac{de'}{s^3}\left[\frac{\vec{r}}{c^2} \times \left\{ \left(\vec{r} - \frac{r\vec{u}}{c}(t') \right) \times \dot{\vec{u}}(t') \right\} + \left\{ 1 - \left(\frac{\vec{u}(t')}{c} \right)^2 \right\} \left(\vec{r} - \frac{r\vec{u}(t')}{c} \right) \right] . \qquad (5.3\text{-}1)$$

This problem cannot be solved without assumption (2) and then the functions of $t' = t - \frac{r}{c}$ may be expanded in powers of $\frac{r}{c}$. Retaining terms up to $(\frac{r}{c})^3$, and integrating over the spherical charge distribution, one gets equation (5.2-6) with

$$m_{el} = \frac{4}{3}\frac{U_0}{c^2}, \tag{5.3-2}$$

where $U_0 \simeq \frac{e^2}{4\pi\epsilon_0 a}$. Two other features of this derivation may be noted.

a) If one retains higher order terms, then third and higher time derivatives of $\vec{u}$ appear in the radiation reaction force and hence in the equation of motion.

b) The higher order terms are proportional to powers of a and vanishing a (point electron) would make the higher terms ineffective at the cost of infinite U_o and hence infinite m_{el}.

5.4 Properties of the Equation of Motion:

Writing m for bare mass plus m_{el}, the equation of motion for an electron becomes

$$\vec{F} = \dot{\vec{p}} - \frac{2\alpha\hbar}{3mc^2}\ddot{\vec{p}} \tag{5.4-1}$$

For $\vec{F} = 0$, this equation has runaway solutions whose time dependence is like $e^{(3t/2\tau)}$. To avoid such a solution, the 'procedure' adopted is to multiply the solution of equation (5.4-1) with an integrating factor $e^{-(3t/2\tau)}$ to obtain

$$\dot{\vec{u}}(t) = \frac{3}{2\tau}\int_t^\infty \frac{\vec{F}(t')}{m}\, e^{-3(t'-t)/(2\tau)}\, dt' . \tag{5.4-2}$$

This solution violates causality for times of order τ because the acceleration at time t is determined by the force that would act in future, till about time τ after t.

5.5 Comments on Radiation Reaction: The Breakdown Limit

So far we have discussed the radiation reaction theory as it existed before quantum theory came to exist. The difficulties with this theory were clear. Among other things, the 'theoretical' electron was not at peace with its own field (but the actual electron never had any such problem!!) It would run away, have infinite mass, violate causality or do many similar things. Not that the equations described above do not work. They work beautifully except at very small distances ($\sim 10^{-13}$ cm) or very small times ($\sim 10^{-23}$ sec.). In fact, radiation reaction can often be neglected. As an example, consider an electron in one dimensional motion with 'constant' acceleration a, starting from rest at time t = 0. After a time t, it has the kinetic energy

$$E_k = \frac{1}{2} m a^2 t^2 \tag{5.5-1}$$

and the energy lost by it through radiation is

$$E_r = \frac{2}{3} m \tau a^2 t \ . \tag{5.5-2}$$

Thus $$(E_r / E_k) \sim (\tau / t) , \tag{5.5-3}$$

which is negligible for $t \gg \tau$. In practice τ is so small that this is often the case.

I would like to pose the following question here: What happens for $t < \tau$? The above equations tell us that the radiated energy is too much compared to the kinetic energy. The external agency is giving too much energy to radiation fields. But then the effect of these fields would be a violent reaction on the electron which can damp (or overdamp) its motion a great deal. This question may be brushed aside by saying that we do not care for such time intervals about which we have additional doubts. But every motion starting at $t = 0$ would pass through this period of time about which the above equations give an absurd picture. The correct answer seems to be that equations (5.5-1) and (5.5-2) break down for $t \sim \tau$ and the above mentioned difficulty is related to the difficulties of radiation reaction force. In fact, there is an alternative physical picture. An electron (May God bless it with peace) is sitting along with (at least) its induction fields surrounding the particle. Now we try to accelerate it and in this process it radiates and decelerates. So for sometime we cannot even get the desired acceleration and <u>during this small interval of time, the time derivatives of momentum $\vec{p}$ are dependent on $\vec{p}$ and related to each other</u>. Such a picture is a radical departure from the Newtonian notion of a constant external force. But it must be noted that this departure is for a short time: in fact equation (5.5-3) is also the ratio of energy pumped into the field and the energy gained by the particle. Newtonian ideas hold after particle energy has become more than what is given to the self fields.

The above type of consideration will appear many times in the rest of these lectures. There are other independent indications of the limit for the breakdown of usual electrodynamics. For example, the classical Thompson scattering (scattering of light of frequency w by electrons) fails when $(\hbar w/c) \gtrsim mc$ or $hw \gtrsim mc^2$ (and it goes over to Compton scattering). This is also the case of field energy $\hbar w$ exceeding the particle energy mc^2. There are many other examples, some of which will be discussed later on.

Numerous attempts have been made to develop a satisfactory classical theory of electrons, and there are claims that such formulations exist in some sense. But the reality is that small problems like Lorentz covariance etc. have been solved while the more serious problems of diverging self-energy and violation of causality have been dumped in other things like bare mass, advanced potentials etc., which have some make belief arguments like substraction procedures, asymptotic formulations etc. These coupled with the uncertainties of the quantum aspects of the problem and the mysteries of breakdown of physical laws for small distances and small time intervals have sufficed to sink the basic unsolved problems in a sea of general confusion.

I would like to propose an entirely different but simpler point of view which will lead us to the Schrödinger theory. The particle called electron is always surrounded by electromagnetic field. Any motion of this particle plus field entity must affect the properties of both the constituents. Energy can reside in both but depending upon the state of motion, more energy may reside in one constituent compared to the other. In fact, when energy in the particle is very large compared to that in the field, the particle behavior persists and usual classical laws work including the radiation reaction forces mentioned above. This breaks down when the field has more energy than the particle. There will be radical departure from Newtonian behavior and I call it the quantum regime. Even the radiation reaction force will change in this regime, there will be new types of correlations between the various time derivatives of velocity and new kinds of bound states will appear with discrete energy spectrum.

6.1 Axiom on Radiation Reaction

Here I wish to depart from the usual classical Electrodynamics and propose an entirely new physical picture. This is based on two basic points mentioned earlier:

1) For small distances and short times, the usual ideas of electrodynamics do not apply. A more precise statement of this breakdown limit is that when the field of the particle (self-field) has energy comparable to the energy of the particle, the usual equations of classical electrodynamics are no more valid.

2) As soon as a force is applied to an electron, the Newtonian mechanics does not hold for a short interval of time. In particular, the transition from $\dot{u} = 0$ to a non zero $\dot{u}$ passes through a small period of time during which the various time derivatives of u are related to one another. Also the inertia or mass renormalization changes a great deal during this period.

One might ask : If the usual Newtonian Mechanics and Electrodynamics do not work here, how do we describe the motion? Yes, the motion is described by a new mechanics and let us call it the Microscopic Mechanics. But we know an alternative answer. It is Quantum Mechanics which '<u>must</u>' describe the motion. As we shall see, Microscopic Mechanics is closely related to the Schrödinger's Quantum Theory.

A more radical physical picture is that in atomic systems the motion may indeed be <u>very slow</u> in contrast to the usual belief of violent motions taking place in atomic systems. Do we believe in the velocity c for Dirac electrons and αc for the electron of hydrogen atom in the first Bohr orbit? The picture of a planetary atom was needed to balance the force of attraction between electron and nucleus. But an extra force appears in Microscopic Mechanics to do this job. I think it is not impossible to imagine that much of the motions in atomic dimensions can be slow, and may belong to the type of motion not describable by usual electrodynamics.

Now the basic problem is to 'derive' the 'correct' radiation reaction force. Many derivations exist, each claiming to have overcome some fundamental difficulty but I believe none of these is 'completely satisfactory'. However some important points emerge from such derivations. For example, if an extended electron is assumed with non-rigid charge distribution, then product of $\dot{p}$, $\ddot{p}$ and $\dddot{p}$ appear in F_r. Note that these time derivatives of p may be interrelated when the effect of F_r is large. Also (e^2/c) and hence $\hbar$ will appear in F_r. Instead of attempting 'yet another derivation' of F_r, I propose the following axiom for radiation reaction which can take us deeper into the physical picture described so far:

<u>Axiom IV:</u> The radiation reaction force should be such that

a) the resulting equation of motion is connected with the quantum equations,

b) the expression reduces to the usual expression in the proper limit, and

c) the classical mechanics is also obtained in the proper limit.

A damping force is usually quite complicated but a damping force required to satisfy the above conditions is expected to be even more complicated. Anyway, such a force can be written down so that the resulting equation of motion is connected with the Schrödinger's equation. (I must emphasize here that often the energy expressions are sufficient and it is not necessary to write down the force explicitly). As an example, consider an electron that moves along one direction, say x-axis. The radiation reaction force may be written down as

$$F_r = \sqrt{(1-\frac{u^2}{c^2})}\left(\frac{m^2\hbar^2}{4}\right)\left[\frac{8\dot{p}\ddot{p}}{p^5} - \frac{10(\dot{p})^3}{p^6} - \frac{\dddot{p}}{p^4}\right] \tag{6.1-1}$$

Now I proceed to show that this force satisfies all the above requirements. To do so, consider the equation of motion under an external force F and expand the factor $\sqrt{(1-u^2/c^2)}$ as $[1-p^2/(2m^2c^2)]$ to get

$$F = \dot{p} - F_r = \dot{p} - \left(1-\frac{p^2}{2m^2c^2}\right)\frac{m^2\hbar^2}{4}\left[\frac{8\dot{p}\ddot{p}}{p^5} - \frac{10(\dot{p})^3}{p^6} - \frac{\dddot{p}}{p^4}\right]. \tag{6.1-2}$$

It is clear that for large p, F_r is negligible (also see below). However, one cannot neglect F_r when it is in some way comparable with p and in that regime, the other time derivatives of p may be related to p (see comments on equation (5.1-7) and after equation (5.5-3)). Then each term in F_r is proportional to $\dot{p}$. The simplest term to consider is

$$[(m^2\hbar^2\dot{p}^3)/p^6] \propto \dot{p} \, ,$$

whence

$$\dot{p} = \beta p^3/(m\hbar) \, , \tag{6.1-3}$$

with a numerical constant β. This relation is consistent with other terms of F_r because

$$\ddot{p} = \frac{3\beta p^2 \dot{p}}{m\hbar} = \frac{3\dot{p}^2}{p} \, , \quad \dddot{p} = \frac{6\dot{p}\ddot{p}}{p} - \frac{3\dot{p}^3}{p^2} \, . \tag{6.1-4}$$

(Note that $\ddot{p}p \propto \dot{p}^2$ like in equation (5.1-7)).
Then equation (6.1-2) becomes

$$F = \dot{p} - \left(1 - \frac{p^2}{2m^2c^2}\right)\frac{m^2\hbar^2}{4}\left[\frac{8\dot{p}\ddot{p}}{p^5} - \frac{10\dot{p}^3}{p^6} - \frac{6\dot{p}\ddot{p}}{p^5} + \frac{3\dot{p}^3}{p^6}\right]$$

$$= \dot{p} - \left(1 - \frac{p^2}{2m^2c^2}\right)\frac{m^2\hbar^2}{4}\left[\frac{2\dot{p}}{p^5}\ddot{p} - \frac{7\dot{p}^3}{p^6}\right]$$

$$= \dot{p} - \left(1 - \frac{p^2}{2m^2c^2}\right)\frac{m^2\hbar^2}{4}\left[\frac{2\dot{p}\ddot{p}}{p^5} - \frac{7}{3}\frac{\dot{p}\ddot{p}}{p^5}\right]$$

$$= \dot{p} - \left(1 - \frac{p^2}{2m^2c^2}\right)\left(-\frac{m^2\hbar^2}{12}\frac{\dot{p}\ddot{p}}{p^5}\right)$$

$$= \dot{p} + \frac{m^2\hbar^2}{12}\frac{\dot{p}\ddot{p}}{p^5} - \frac{\hbar^2}{24c^2}\frac{\dot{p}\ddot{p}}{p^3}$$

or $$F = \dot{p} + \frac{\beta^2}{4}\dot{p} - \frac{\hbar\beta}{24mc^2}\ddot{p} \qquad (6.1\text{-}5)$$

For $\beta = 16\alpha \simeq 0.11676$, the equation of motion (6.1-5) reduces to equation (5.4-1) with Abraham-Lorentz force. Then the mass renormalization is only $\beta^2/_4 \sim 0.0034$ or 0.34 percent. This is a perfectly reasonable way of satisfying condition (b) of the axiom.

6.2 Connection with Schrödinger's Equation

For u << c, F_r of equation (6.1-1) leads to the equation of motion

$$F = \dot{p} - \frac{m^2\hbar^2}{8p}\frac{d}{dt}\left[\frac{5\dot{p}^2}{p^4} - \frac{2\ddot{p}}{p^3}\right] . \qquad (6.2\text{-}1)$$

This may be written as

$$F = \dot{p} - \frac{m\hbar^2}{8}\frac{d}{dx}\left[\frac{5(\dot{p})^2}{p^4} - \frac{2\ddot{p}}{p^3}\right] , \qquad (6.2\text{-}2)$$

where the relation (d/dx) = (m/p)(d/dt) has been used. Multiply equation (6.2-2) by dx and introduce external potential V(x) due to F to get

$$E = \frac{p^2}{2m} + V(x) + \frac{m\hbar^2}{8}\left[\frac{2\ddot{p}}{p^3} - \frac{5\dot{p}^2}{p^4}\right] \tag{6.2-3}$$

$$= \frac{p^2}{2m} + V(x) + \frac{\hbar^2}{2m}\left[\frac{p''}{2p} - \frac{3}{4}\left(\frac{p'}{p}\right)^2\right] \tag{6.2-4}$$

$$= \frac{p^2}{2m} + V(x) + \frac{\hbar^2}{2m}\left[\frac{d}{dx}\left(\frac{p'}{2p}\right) - \left(\frac{p'}{2p}\right)^2\right] \tag{6.2-5}$$

where we have used the relation $p' = dp/dx = \frac{m}{p}\frac{dp}{dt} = m\dot{p}/p$.

Now define two real functions W(x) and A(x) through the relations

$$p = W' = \frac{d}{dx} W(x) \quad , \tag{6.2-6}$$

and

$$A'(x) = p'/(2p) \quad , \tag{6.2-7}$$

so that equation (6.2-5) becomes

$$E = \frac{(W')^2}{2m} + V(x) + \frac{\hbar^2}{2m}\left[A'' - (A')^2\right] . \tag{6.2-8}$$

Finally, a complex function ψ may be defined as

$$\psi(x) = L^{-\frac{1}{2}} \exp\left[-A(x) + iW(x)/\hbar\right] \quad , \tag{6.2-9}$$

with a (normalization) constant L. Then equations (6.2-8) and (6.2-7) are respectively the real and imaginary parts of the Schrödinger's equation

$$\psi''(x) + \frac{2m}{\hbar^2}(E-V)\psi(x) = 0. \qquad (6.2\text{-}10)$$

Thus the equation of motion (6.2-1) is connected with the corresponding quantum equation and requirement (a) of the Axiom is satisfied. Note that the same total energy E and potential energy V(x) enter the Schrödinger's equation as the corresponding equation (6.2-3) of the Microscopic Mechanics. But a completely new physical picture has emerged through the above connection.

6.3 Mathematical and Physical Aspects of the Connection

The procedure followed above for connecting the equation of motion (6.2-1) with the Schrödinger's equation is fairly simple but originally this was found using an involved mathematical reasoning. I wish to mention this because that also provides insights into some mathematical aspects of this problem. It concerns a class of non-linear equations, e.g. the so called KdV non-linear classical wave equation; which have stable solutions called 'solitons'. These are quite fashionable objects in many current investigations in theoretical physics. A general non-linear equation may not have stable solutions (Heisenberg once stated that during a simulation of a particle trajectory in an accelerator using a non-linear equation, it was found that the trajectory suddenly becomes unstable after going round smoothly more than 2000 times!!)
Then, what are the criteria for a given non-linear equation to have soliton solutions?

The following answer has been provided for the above question. A given non-linear equation will definitely have stable (soliton) solutions if the equation can be mapped into linear differential operators (I would like to emphasize that such an existence proof does not involve one to one reciprocal mapping). This led many people to seek a non-linear wave equation corresponding to Schrödinger's linear differential operator. I sought the correspondence with a non-linear equation of motion which turned out to be equation (6.2-1). Such is the mathematical foundation of the connection between the equations (6.2-1) and the Schrödinger's equation (6.2-10). An important implication of this is that there are mathematical reasons for the existence of some well behaved and stable solutions of the proposed equation of motion (equation (6.2-1)).

The physical aspects of the connection between (6.2-1) and (6.2-10) have more interesting implications. If we neglect F_r in equation (6.2-1) or simply set $\hbar \to 0$ (which I do not like because $\hbar$ is a non-zero constant), then the end equation is equation (6.2-8) without its last term. This is just the Hamilton-Jacobi equation (1.2-9). Therefore equation (6.2-8) is an 'extended Hamilton-Jacobi equation' which has resulted from the new equation of motion (6.2-1). The corresponding mathematically equivalent ψ waves have the amplitudes and phases related by this extended Hamilton - Jacobi type of equation. Indeed, a further step is to take

$$\begin{aligned} p(x) &= \psi^*(x)\,\psi(x) \\ &= \frac{1}{L}\exp\left[-2A(x)\right] \end{aligned} \qquad (6.3\text{-}1)$$

But from equations (6.2-6) and (.2-7) one gets

$$A' = \frac{p'}{2p} = \frac{1}{2}\frac{d}{dx}(\ln p)$$

whence $$A(x) = \frac{1}{2}\ln(p/p_0)\,, \qquad (6.3\text{-}2)$$

where p_o is some suitable constant value. Then equation (6.3-1) yields

$$P(x) = \frac{p_o}{L\,p} \qquad (6.3\text{-}3)$$

This is clearly the classical probability density because $P(x)dx \propto (\frac{dx}{v})$, gives the time spent by the particle in the interval dx. Therefore, $\psi(x)$ is the probability amplitude for finding the particle between x and x+dx (when such an reinterpretation is really needed). A simple paradox arises in using equation (6.3-3) and $\psi^*(x)\psi(x)$ in the case of Harmonic oscillator which will be discussed later on.

Another important physical aspect of the connection lies in equation (6.2-3) which has the extra energy term

$$E_r = \frac{m\hbar^2}{8p^2}\left[\frac{2\ddot{p}}{p} - \frac{5\dot{p}^2}{p^2}\right] \qquad (6.3\text{-}4)$$

corresponding to F_r in the equation of motion. This extra energy which resides in the field as a mass renormalization effect is very small for the Newtonian kind of motions. Therefore, the corresponding Hamilton-Jacobi equation does not lead to Schrödinger's equation. This provides the clearest separation between classical regime and Quantum regime. It also tells us that particle energy and therefore particle behavior is dominant in the classical regime. E_r is inversely proportional to mass and hence should be negligible for massive charged particles. It is an experimental fact that ionic motions are very often adequately described by classical Mechanics. For $\hbar \to 0$, the classical limit is also reached. In fact, equation (6.2-3) incorporates every known regime of classical and quantum behavior except that the momentum dependence is new. It also shows that Schrödinger's equation yields energy values that include E_r.

6.4 Limitations of Quantum Concepts

We do not intend to discuss most of the well known objections against quantum theory which have now been turned into attempts at understanding the origin of quantum behavior at some curious mathematical level (in contrast with the simple physical picture provided above). Some new kinds of simple reasoning will be presented here to justify (a) the correct mathematical results, (b) some correct physical pictures, together with (c) the possibility of misleading physical interpretations in the framework of quantum theory. To do so, a very brief criticism of quantum theory should suffice.

It is absolutely necessary to recognize that the act of observation on microscopic systems produces large disturbances so that the uncertainty principle holds and the correct statistical results are provided by quantum theory. But there have been many deviations from this basic idea and some of them have even been misleading in the physical sense. Often the end result has been a mix up of quantum results and classical ideas and some honest people like to call this a semi-classical procedure. The other aspect of the story is that the Schrödinger's single particle theory leads to the single particle relativistic Dirac electron which must travel with speed c (but relativity allows speed c for massless paticles only!). In addition, the infinite sea of negative energy electrons and the negative probability density for Schrödinger's relativistic equation appear in the physical picture. The logical extension into the many particle field theory gives dressed particles with divergent renormalization terms whose parallels have already existed in classical electrodynamics.

The inverted pyramids of additional quantum numbers, sometimes of mysterious physical origin has characterized the major efforts at understanding the high energy experiments. I am of the opinion that many times the distinction between the mathematical equivalence and the physical reality has been lost and this leads to distorted physical picture. The simple example of virtual processes can illustrate this point. One kind of virtual processes can arise due to the use of complete sets of states in evaluating expectation values of the products of operators (as for example in perturbation theories). Such mathematically equivalent procedures never imply that such processes should be really taking place in nature. The other kind of virtual processes are those where the uncertainty principle is invoked to imagine that certain unobservable physical processes are taking place in nature. Whereas such processes <u>may</u> take place in nature, there should remain the possibility of identifying the various physical sources. For example, if a particle B throws out a virtual particle b to recapture it within a time interval such that the processes cannot be observed, it is desirable to identify the source which supplies energy for the creation of b. This fine distinction between possibility and actual occurance is often ignored in quantum theory and I consider it a serious limitation of quantum concepts.

One final remark about the radiative and mass renormalization effects is that these have received rather arbitrary treatment in quantum theory. No convincing reason is ever advanced for the non-radiative treatment of so many problems in many branches of physics. This becomes really awkward when the internal fields in some solids for example, assume unusually high values implying very large accelerations and radiative corrections. Even the simpler (?) mass renormalization term is almost always missing. One has two alternative answers

1) The radiative corrections and mass renormalization terms are always very very small or,

2) some of the radiative and mass renormalization effects are already included in quantum results. Here we adopt this second point of view.

7.1 Microscopic Mechanics - A Causal Microscopic Theory

The new equation of motion (6.2-1), claimed to describe the motion of the electron surrounded by the self fields, departs from the usual notions of Newtonian Mechanics when the electron starts from rest or comes to rest. This behavior is confined to a small time interval and also quite simple to understand. In the process of achieving a constant acceleration, one has to build up velocity, acceleration etc. from a zero value and for a short (indeed very short, say 10^{-20} sec) interval of time, the various time derivatives of velocity are related to one another. This unfolds the new microscopic mechanics. Many aspects of this new mechanics are yet to be investigated and clarified. I will present some of the aspects which seems to be clear at present. Many of the arguments presented will be intuitive and physical; and rigorous mathematical excercises will be avoided. The basic reason for this is the fact that the Schrödinger's equation yields the observable quantities with far greater mathematical ease than the complicated equation of motion in Microscopic mechanics. The latter however can clarify the physical picture, clearly separate the classical and quantum behavior and shows an outlet from the 'we do not know' enclosure of quantum concepts.

It is interesting to note that following the famous objections of Einstein against Quantum Theory, Bohm and Vigier (1954) had derived equation (6.2-5) from an entirely different physical picture. They postulated that an electron is a particle with a well defined trajectory accompanied by a physically real wave field ψ. This field was assumed to have three specific properties (a) ψ satisfies Schrödinger's equation (b) momentum of the electron is proportional to the gradient of the phase of ψ and (c) the probability distribution of an ensemble of electrons with the same ψ is given by $|\psi|^2$. This last assumption was so strange that this work was burried under its criticism. We have seen that in the present approach, the three assumptions (a), (b) and (c) of Bohm and Vigier are results but ψ is not the physically real wave field surrounding the electron. Indeed, the physically real wave field that surrounds the electron is its self-field and then the axiom on the self-force leads to the properties of ψ.

It shows that Microscopic Mechanics provides a description of the motion of an isolated (not from self-fields!!) electron and also tells us that when observations are made on an ensemble of such systems, the probabilities for the results are provided by Schrödinger's theory. Many attempts exist in literature to postulate (or prove the non-existence of) such a mechanics whose statistical behavior will be given by quantum theory. The mere existence of Microscopic Mechanics as described above is perhaps the simplest of all such successful attempts.

7.2 Harmonic Oscillator

Let us consider the problem of a one-dimensional harmonic oscillator with

$$V(x) = \frac{1}{2} m \omega^2 x^2 \qquad (7.2\text{-}1)$$

where ω is the classical frequency. The equation of motion of the electron, according to microscopic mechanics is

$$-m\omega^2 x = \dot{p} - \frac{m^2\hbar^2}{4}\left(\frac{8\dot{p}\ddot{p}}{p^5} - \frac{10(\dot{p})^3}{p^6} - \frac{\dddot{p}}{p^4}\right) \qquad (7.2\text{-}2)$$

The corresponding energy expression is (from equation (6.2-5))

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \frac{\hbar^2}{2m} \left[\frac{d}{dx} \left(\frac{p'}{2p} \right) - \left(\frac{p'}{2p} \right)^2 \right] \qquad (7.2\text{-}3)$$

A) In the classical or Newtonian approximation, the last term is neglected and one gets

$$p = \sqrt{(2mE - m^2 \omega^2 x^2)} \qquad (7.2\text{-}4)$$

The corresponding probability density is

$$P_c(x) = \frac{p_0}{L\,p} = \frac{p_0}{L \sqrt{(2mE - m^2 \omega^2 x^2)}} \, . \qquad (7.2\text{-}5)$$

B) In the Larmor approximation (7.2-3) would have its last term replaced by a suitable integration over energy loss. Actually, the equation (7.2-2) of motion is given an Abraham-Lorentz force (5.2-6)

$$- m \omega^2 x = \dot{p} - \frac{2}{3} \tau \ddot{p} \quad , \qquad (7.2\text{-}5)$$

with τ given by equation (5.2-7). This damping force is small. Therefore one uses $p = -m\omega^2 x$ to get $\ddot{p} = -\omega^2 p$ and then equation (7.2-5) becomes

$$\dot{p} + m \omega^2 x + \frac{2}{3} \omega^2 \tau p = 0 \; . \qquad (7.2\text{-}6)$$

Again, the last term is small. The damped oscillator has its position x(t) and momentum p decay as $exp[-\frac{2\omega^2 \tau}{3} t]$. As time goes on, the oscillator radiates and finally stops oscillating. The probability density $P_c(x)$ of equation (7.2-5) which had two singularities at the turning points changes with time and finally the two singularities merge into one at the origin.

(C) In the Quantum regime, the full equation of motion (7.2-2) must be retained. The damping force is neither negligible nor small. But still it is <u>a damping force which cannot cause motion on its own</u>. (Note that this physical requirement has <u>never</u> been used for the Abraham-Lorentz damping force or its generalizations). Therefore, the damping force cannot exceed the external force. <u>As a limiting case, the damping force can be as large as the applied force itself</u>.

The F_r in Microscopic Mechanics becomes large as p becomes small. It would have become very large indeed as $p \to 0$, but as we have repeatedly mentioned, all the other time derivatives of p are related to p for small p motion and in the limit, $\dot{p}$, $\ddot{p}$ and $\dddot{p}$ also tend to zero as p tends to zero. Then, setting $\dot{p} \to 0$ in equation (7.2-2), we get the equation of critical damping as

$$-m\omega^2 x = \lim_{(p,\dot{p},\ddot{p},\dddot{p}\ldots)\to 0} \left[-\frac{m^2\hbar^2}{4}\left(\frac{8\dot{p}\ddot{p}}{p^5} - \frac{10(\dot{p})^3}{p^6} - \frac{\dddot{p}}{p^4} \right) \right] \qquad (7.2\text{-}7)$$

which is only the equality $F = -F_r$ in the limit of p, $\dot{p}$, $\ddot{p}$ .... tending to zero.

Let

$$y = \frac{p'}{2p} \qquad (7.2\text{-}8)$$

where p' is $\frac{dp}{dx}$. Then equation (7.2-7) can be written as

$$\frac{2m^2\omega^2 x}{\hbar^2} = \frac{d}{dx}\left(y^2 - y' \right) . \qquad (7.2\text{-}9)$$

The solution of this equation for y as a function of x should be used in the energy expression (7.2-3) which for $p \to 0$ can be written as

$$E = \frac{1}{2} m\omega^2 x^2 + \frac{\hbar^2}{2m}\left[y' - y^2\right] \tag{7.2-10}$$

A quick look at equations (7.2-9) and (7.2-10) shows that E must be constant. To get that constant and to derive further information, we can solve equation (7.2-9) for y. This is not easy because stable solutions exist only for certain discrete energy values. But it is very simple to check some of these solutions. Consider for example the following two solutions

$$y_0 = \frac{m\omega}{\hbar} x \quad , \tag{7.2-11}$$

and

$$y_1 = \frac{m\omega}{\hbar} x - \frac{1}{x} \quad . \tag{7.2-12}$$

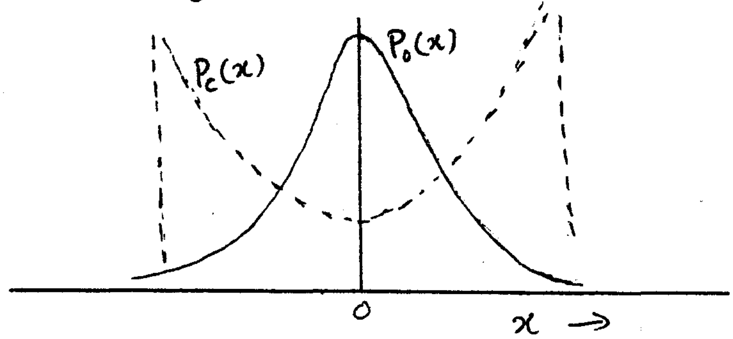
These solutions give $E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$ respectively. It should be clear that all the energy levels obtained from the Schrödinger's equation for Harmonic oscillator can be reproduced in this way. Even the wave functions $\psi(x)$ can also be obtained exactly. Thus, from (7.2-11), we can integrate to get

$$P_0(x) = \frac{p_0}{Lp} = \frac{1}{L} \exp\left[-\frac{m\omega x^2}{\hbar}\right] = \psi^*(x)\psi(x) \tag{7.2-13}$$

so that

$$\psi_0(x) = L^{-\frac{1}{2}} \exp\left[-\frac{m\omega}{2\hbar} x^2\right] \quad . \tag{7.2-14}$$

This is the ground state wave function. We have taken the square root of equation (7.2-13) because for $p \to 0$, $W(x)$ is a constant phase which can always be chosen to be zero. Hence the wave function is real. Similarly, equation (7.2-12) can give us $\psi_1(x)$ for the next energy state. It is interesting to note that the <u>same basic definition of probability density</u> gives us $P_c(x)$ in equation (7.2-5) and $P_o(x)$ in equation (7.2-13). $P_c(x)$ is obtained in the approximation of classical mechanics. In the standard textbooks on Quantum theory, the almost opposite characteristics of $P_c(x)$ and $P_o(x)$ is emphasized to show that the probability density $\psi^*(x)\psi(x)$ <u>is not defined</u> in the same sense as the classical probability density $P_o(x)$. But one continues to use this mysterious probability density $\psi^*(x)\psi(x)$ in the ordinary sense and also for large energy values it '<u>somehow</u>' goes over to $P_o(x)$!



Microscopic Mechanics yields a different picture. The basic definitions of $P_c(x)$ and $P_o(x)$ <u>are the same</u>. An electron with undamped oscillations (if at all possible!) would have $P_c(x)$ and continue to oscillate indefinitley. But a damped motion of the oscillating electron must bring it to a stop. It is most probable that the electron will stop at x = 0. We can go very deep into this problem but I wish to stop here with only one additional point. There is a new type of damping force in Microscopic Mechanics as compared to classical mechanics. This leads to a new class of bound states that are usually not found in classical mechanics, and the ground state of an electron subjected to a simple Harmonic force is an example of this kind of bound state. There is no oscillation at all.

The self fields surrounding the electron has modified its inertia such that there is no oscillation at all. The energies of this new class of bound states is given by Schrödinger's quantum theory.

Hydrogen atom problem can be done and has been done in a similar way. The known results are reproduced but the physical pictures are very different. The electrons surrounded by self-fields and assuming different (but finite) mass renormalizations under different external conditions are moving not very fast and indeed forming motionless ground states. The picture of static chemical bonds reappears. Radiations are given out from the surrounding fields due to accelerations. Many aspects of such a physical picture are fitting into each other like a jigsaw puzzle but the main point is that it is a different picture than the usual one.

7.3 The New Physical Picture and Open Questions:

The new physical picture provided by Microscopic Mechanics apparently has to face half a century of most active period in theoretical physics. But actually it should suffice to check that it is consistent and fits in at the important places. Many such possibilities exist and I have checked only a few of them. The most satisfying feature of this picture is perhaps the fact that no strange, out of the world concepts are needed to be invoked and there are no divergences, violations of cause and effect relationships *etc*. The motions in atomic systems do not appear to be more violent than what we are capable of producing in the Laboratory. No unobservable and yet divergent processes are supposed to be taking place 'somehow' in the atomic systems. The whole thing appears to have the peace as well as the violence (sometimes large but never perpetually infinite) of nature around us.

Such a picture may not suit the taste of the 'hard core' quantum physicists who would always prefer to have the mysterious, magical possibilities allowed in quantum theory. I would like to pose the following question here: How much of the strict quantum picture has been adhered to in the last half century? The most probable answer is - "very little". There are countless examples where a classical vector is drawn for Angular momentum but quantum theory admits knowledge of its magnitude and only one of the three cartesian components in a given state. The mix up of classical notions with the 'pure' quantum notions is so commonly done that it is difficult to find examples in real life problems where this is avoided. We really have a strange "Eintopf" of quantum picture, classical picture and approximations of various kinds. Such an approach, while good enough for many individual problems, can hardly lead to a unified picture.

The above criticism may lead us to the counter question: Do we at all need Microscopic Mechanics? This is an open question which also leads to many other open questions. Only some partial answers can be given. (1) As we have seen, the existence of microscopic mechanics provides a sort of continuity from Newtonian ideas to Schrödinger's quantum theory. (2) Microscopic mechanics fill up (at least partially) the important gap in our understanding of physics at short distances and short times. (3) It provides the clearest separation of classical and quantum descriptions. (4) Existence of Microscopic Mechanics reinforces the validity of quantum results while at the same time the criteria of physical reality of microscopic systems are preserved. (5) The physical picture has the simple physical basis of the eternal inseparability of particles and fields. (6) It is potentially useful in constructing better and more detailed physical models, especially in view of the evident failure of quantum approaches in high energy physics. Indeed, a new level of physics may be revealed leading to a better understanding of the foundations of theoretical physics.

I have tried to present here a somewhat unified picture of theoretical physics, using seven Axioms (including three for Newtonian Mechanics) to reach quantum theory from classical Mechanics via Classical Electrodynamics. This has required the introduction of Microscopic Mechanics in which only a beginning has been made. It would have served its purpose if this motivates some physicists to persue the quest for a new level of theoretical physics.

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